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VOLUME II.

CONTAINING

THE GEOMETRY.

BY T. S. DAVIES, ESQ., F.R.S., L. & E.;

But in consequence of the Death of Mr. DAVIES, the Volume has been Printed under
the direction of, and the Chapter on CONIC SECTIONS supplied

BY STEPHEN FENWICK, ESQ.

LONDON:
PRINTED FOR HER MAJESTY'S STATIONERY OFFICE:
PUBLISHED BY
HN WEALE, 59 HIGH HOLBORN.
1853.

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Transferred from

The Lawrence Scientific School,

9 February, 1888.

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THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

1. A **POINT** is that which hath no parts, or which hath no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

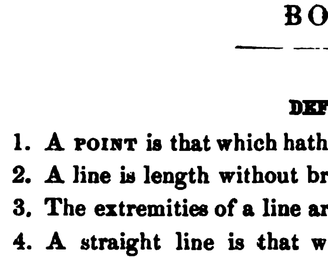
5. A superficies is that which hath only length and breadth.

6. The extremities of a superficies are lines.

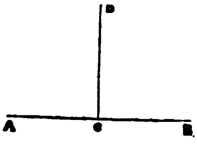
7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

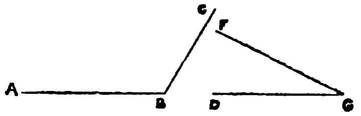


When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line: thus the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; that which is contained by AB, DB, is named the angle ABD, or DBA; and that which is contained by DB, CB, is called the angle DBC, or CBD; but, if there be only one angle at a point, it may be expressed by a letter placed at that point: as the angle at E.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.

11. An obtuse angle is that which is greater than a right angle.

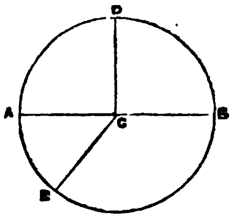


12. An acute angle is that which is less than a right angle.

13. A term or boundary is the extremity of anything.

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.



16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A segment of a circle is the figure contained by a straight line, and the circumference it cuts off.

20. Rectilineal figures are those which are contained by straight lines.

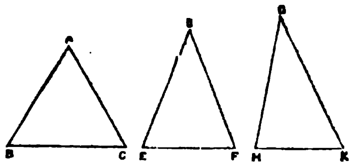
21. Trilateral figures, or triangles, by three straight lines.

22. Quadrilateral by four straight lines.

23. Multilateral figures, or polygons, by more than four straight lines.

24. Of three-sided figures, an equilateral triangle is that which has three equal sides.

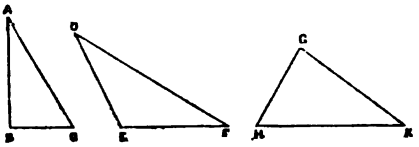
25. An isosceles triangle is that which has two sides equal.



26. A scalene triangle is that which has three unequal sides.

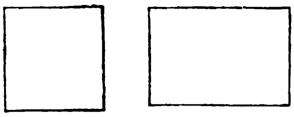
27. A right-angled triangle is that which has a right angle.

28. An obtuse-angled triangle is that which has an obtuse angle.



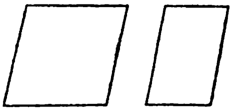
29. An acute-angled triangle is that which has three acute angles.

30. Of four-sided figures, a square is that which has all its sides equal, and all its angles right angles.



31. An oblong is that which has all its angles right angles, but has not all its sides equal.

32. A rhombus is that which has all its sides equal, but its angles are not right angles.



33. A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

34. All other four-sided figures besides these are called Trapeziums.

35. Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

POSTULATES.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

1. Things which are equal to the same are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things which are double of the same are equal to one another.

7. Things which are halves of the same are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

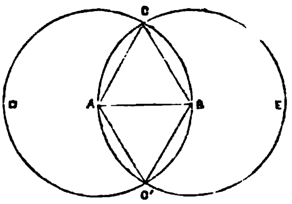
12. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

PROPOSITIONS.

PROPOSITION I.

Problem. *To describe an equilateral triangle upon a given finite straight line.*

Let AB be the given straight line; it is required to describe an equilateral triangle upon it.



From the centre A, at the distance AB, describe (3 *Postulate*) the circle BCD, and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines (1 *Post.*) CA, CB, to the points A, B: ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal (15 *Definition*) to AB; and because the point B is the centre of the circle ACE, BC is equal to BA:

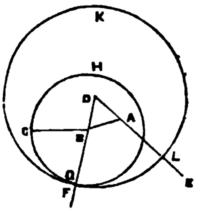
But it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB:

But things which are equal to the same are equal to one another (1 *Axiom*); therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is described upon the given straight line AB. Which was required to be done.

PROPOSITION II.

Prob. *From a given point to draw a straight line equal to a given straight line.*

Let A be the given point, and BC the given straight line; it is required to draw from the point A a straight line equal to BC.



From the point A to B draw (1 *Post.*) the straight line AB; and upon it describe (1. I.) the equilateral triangle DAB, and produce (2 *Post.*) the straight lines DA, DB, to E and F; from the centre B, at the distance BC describe (3 *Post.*) the circle CGH, and from the centre D, at the distance DG, describe the circle GKL: AL shall be equal to BC.

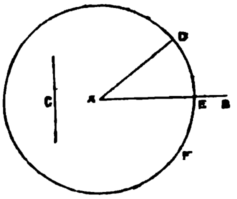
Because the point B is the centre of the circle CGH, BC is equal (15 *Def.*) to BG;

And because D is the centre of the circle GKL, DL is equal to DG, and (*Constr.*) DA, DB, parts of them, are equal; therefore the remainder AL is equal to the remainder (3 *Ax.*) BG;

But it has been shown, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the same are equal (1 *Ax.*) to one another; therefore the straight line AL is equal to BC. Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC. Which was to be done.

PROPOSITION III.

Prob. *From the greater of two given straight lines to cut off a part equal to the less.*



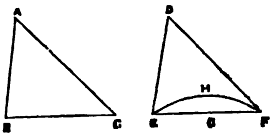
Let AB and C be the two given straight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the less.

From the point A draw (2. I.) the straight line AD equal to C; and from the centre A, and at the distance AD describe (3 *Post.*) the circle DEF;

And because A is the centre of the circle DEF, AE is equal (15 *Def.*) to AD; but the straight line C is likewise equal (*Constr.*) to AD; whence AE and C are each of them equal to AD; wherefore the straight line AE is equal (1 *Ax.*) to C, and from AB, the greater of two straight lines, a part AE has been cut off equal to C the less. Which was to be done.

PROPOSITION IV.

Theorem. *If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; they shall likewise have their bases,* or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite.*



Let ABC, DEF be two triangles, which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the base BC shall be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles to which the equal sides are opposite, shall be equal, each to each, viz., the angle ABC to the angle DEF, and the angle ACB to DFE.

For, if the triangle ABC be applied to DEF, so that the point A may be on D, and the straight line AB upon DE, the point B shall coincide with the point E, because AB is equal (*Hyp.*) to DE; and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal (*Hyp.*) to the angle EDF; wherefore also the point C shall coincide with the point F, because the straight line AC (*Hyp.*) is equal to DF:

But the point B coincides with the point E; wherefore the base BC shall coincide with the base EF, because the point B coinciding with E, and C with F, if the base BC does not coincide with the base EF, two straight lines would enclose a space, which is impossible (10 *Ax.*). Therefore the base BC shall coincide with the base EF, and be equal to it.

Wherefore the whole triangle ABC shall coincide with the whole triangle DEF, and be equal to it (8 *Ax.*); and the other angles of the one shall coincide with the remaining angles of the other, and be equal to them, viz., the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, their bases shall likewise be equal, and the triangles be equal, and their other angles to which the equal sides are opposite shall be equal, each to each. Which was to be demonstrated.

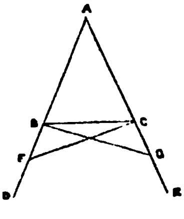
PROPOSITION V.

THEOR. ***The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.***

Let ABC be an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB, AC be produced to D and E, the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater, cut off AG equal to (3. i.) AF, the less, and join FC, GB.

Because AF is equal (*Constr.*) to AG, and AB to (*Hyp.*) AC, the two sides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the base FC is equal (4. i.) to the base GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal (4. i.) to the remaining angles of the other, each to each, to which the equal sides are opposite, viz., the angle ACF to the angle ABG, and the angle AFC to the angle AGB:



And because the whole AF is equal to the whole AG (*Constr.*), of which the parts AB, AC, are equal (*Hyp.*); the remainder BF shall be equal (3 *Ax.*) to the remainder CG; and FC was proved to be equal to GB; therefore the two sides BF, FC are equal to the two CG, GB, each to each; and the angle BFC is equal to the angle CGB;

wherefore the triangles BFC, CGB are equal (4. I.), and their remaining angles, each to each, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG:

And, since it has been demonstrated, that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF are also equal; the remaining angle ABC is therefore equal (3 *Ax.*) to the remaining angle ACB, which are the angles at the base of the triangle ABC. And it has also been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other side of the base.

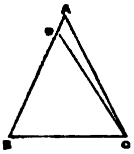
Therefore the angles at the base, etc. Q. E. D.

COROLLARY. Hence every equilateral triangle is also equiangular.

PROPOSITION VI.

THEOR. *If two angles of a triangle be equal to one another, the sides also which subtend*, or are opposite to, *the equal angles, shall be equal to one another.*

Let ABC be a triangle having the angle ABC equal to the angle ACB; the side AB is also equal to the side AC.

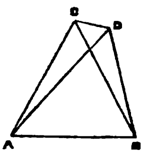


For, if AB be not equal to AC, one of them is greater than the other. Let AB be the greater; and from it cut (3. I.) off DB equal to AC, the less, and join DC; therefore, because in the triangles DBC, ACB, DB is equal to AC (*Constr.*), and BC common to both, the two sides, DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle (*Hyp.*) ACB; therefore the base DC is equal to the base AB, and the triangle DBC is equal to the triangle (4. I.) ACB, the less to the greater; which is absurd. Therefore AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles, etc. Q. E. D.

COR. Hence every equiangular triangle is also equilateral.

PROPOSITION VII.

THEOR. *Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.*

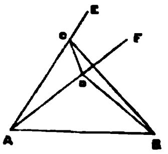


If it be possible, let there be two triangles ACB, ADB, upon the same base AB, and upon the same side of it, which have their sides CA, DA terminated in the extremity A of the base equal to one another, and, likewise their sides, CB, DB, that are terminated in B.

Join CD; then, in the case in which the vertex of each of the triangles is without the other triangle, because AC is equal (*Hyp.*) to AD, the angle ACD is equal (5. I.) to the angle ADC: but the angle ACD is greater (9 *Ax.*) than the angle BCD; therefore the angle ADC is greater also than BCD; much more then is the angle BDC greater than the angle BCD.

Again, because CB is equal (*Hyp.*) to DB, the angle BDC is equal (5. I.) to the angle BCD: but it has been demonstrated to be greater than it; which is impossible.

But if one of the vertices, as D, be within the hour triangle ACB; produce AC, AD to E, F; therefore, because AC is equal (*Hyp.*) to AD in the triangle ACD, the angles ECD, FDC upon the other side of the base CD are equal (5. I.) to one another: but the angle ECD (9. *Ax.*) is greater than the angle BCD: wherefore the angle FDC is likewise greater than BCD; much more then is the angle BDC greater than the angle BCD.



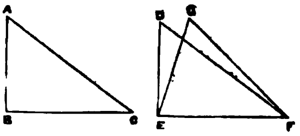
Again, because CB is equal (*Hyp.*) to DB, the angle BDC is equal (5. I.) to the angle BCD; but BDC has been proved to be greater than the same BCD; which is impossible. The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity. Q. E. D.

PROPOSITION VIII.

THEOR. *If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.*

Let ABC, DEF be two triangles, having the two sides AB, AC, equal to the two sides DE, DF, each to each, viz. AB to DE, and AC to DF; and also the base BC equal to the base EF. The angle BAC is equal to the angle EDF.



For, if the triangle ABC be applied to DEF, so that the point B be on E, and the straight line BC upon EF; the point C shall also coincide with the point F, because BC is equal to EF (*Hyp.*).

Therefore BC coinciding with EF; BA and AC shall coincide with ED and DF; for, if the base BC coincides with the base EF, but the sides BA, CA do not coincide with the sides ED, FD, but have a different situation as EG, FG, then upon the same base EF, and upon the same side of it, there can be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity: but this is impossible (7. I.); therefore, if the base BC coincides with the base EF, the sides BA, AC cannot but coincide with the sides ED, DF;

Wherefore likewise the angle BAC coincides with the angle EDF, and is equal (8 *Ax.*) to it. Therefore if two triangles, etc. Q. E. D.

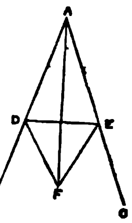
PROPOSITION IX.

Prob. *To bisect a given rectilineal angle, that is, to divide it into two equal angles.*

Let BAC be the given rectilineal angle, it is required to bisect it.

Take any point D in AB, and from AC cut (3. i.) off AE equal to AD; join DE, and upon the side DE remote from A, describe (1. i.) an equilateral triangle DEF; then join AF; the straight line AF bisects the angle BAC.

Because AD is equal (*Constr.*) to AE, and AF is common to the two triangles DAF, EAF; the two sides DA, AF, are equal to the two sides EA, AF, each to each; and the base DF is equal to the base EF (24 *Def.*); therefore the angle DAF is equal (8. i.) to the angle EAF; wherefore the given rectilineal angle BAC is bisected by the straight line AF. Which was to be done.



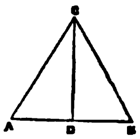
PROPOSITION X.

Prob. *To bisect a given finite straight line, that is, to divide it into two equal parts.*

Let AB be the given straight line; it is required to divide it into two equal parts.

Describe (1. i.) upon it an equilateral triangle ABC, and bisect (9. i.) the angle ACB by the straight line CD. AB is cut into two equal parts in the point D.

Because AC is equal to CB (24 *Def.*), and CD common to the two triangles ACD, BCD; the two sides AC, CD are equal to BC, CD, each to each; and the angle ACD is equal (*Constr.*) to the angle BCD; therefore the base AD is equal to the base (4. i.) DB, and the straight line AB is divided into two equal parts in the point D. Which was to be done.

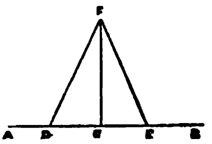


PROPOSITION XI.

Prob. *To draw a straight line at right angles to a given straight line, from a given point in the same.*

Let AB be a given straight line, and C a point given in it; it is required to draw a straight line from the point C at right angles to AB.

Take any point D in AC, and make (3. i.) CE equal to CD, and upon DE describe (1. i.) the equilateral triangle DFE, and join FC, the straight line FC drawn from the given point C is at right angles to the given straight line AB.

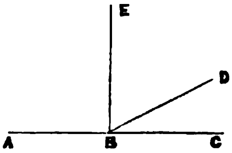


Because DC is equal (*Constr.*) to CE, and FC common to the two triangles DCF, ECF; the two sides DC, CF, are equal to the two EC, CF, each to each; and the base DF is equal to the base EF (24 *Def.*); therefore the angle DCF is equal (8. i.) to the angle ECF; and they are adjacent angles.

But, when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right (10 *Def.*) angle; therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given straight line AB, FC has been drawn at right angles to AB. Which was to be done.

Cor. By help of this problem, it may be demonstrated, that two straight lines cannot have a common segment.

If it be possible, let the two straight lines ABC, ABD have the segment AB common to both of them. From the point B draw BE at right angles to AB; and because ABC is a straight line, the angle CBE is equal (10 *Def.*) to the angle EBA; in the same manner, because ABD is a straight line, the angle DBE is equal to the angle EBA; wherefore (1 *Ax.*) the angle DBE is equal to the angle CBE, the less to the greater; which is impossible; therefore two straight lines cannot have a common segment.

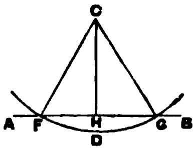


PROPOSITION XII.

Prob. *To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.*

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a straight line perpendicular to AB from the point C.

Take any point D upon the other side of A B, and from the centre C, at the distance CD, describe (3 *Post.*) the circle FDG meeting AB in F, G; and bisect (10. I.) FG in H, and join CF, CH, CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.

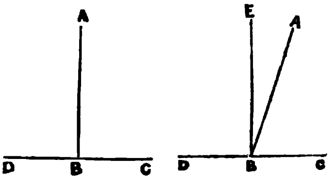


Because FH is equal (*Constr.*) to HG, and HC common to the two triangles FHC, GHC, the two sides FH, HC are equal to the two GH, HC, each to each; and the base CF is equal (15 *Def.*) to the base CG; therefore the angle CHF is equal (8. I.) to the angle CHG; and they are adjacent angles; but when a straight line standing on a straight line makes the adjacent angles equal to one another, each of them is a right angle; and the straight line which stands upon the other is called a perpendicular (10 *Def.*) to it; therefore from the given point C a perpendicular CH has been drawn to the given straight line AB. Which was to be done.

PROPOSITION XIII.

Theor. *The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.*

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD: these are either two right angles, or are together equal to two right angles.



For if the angle CBA be equal to ABD, each of them is a right (10 *Def.*) angle; but if not, from the point B draw BE at right angles (11. I.) to CD; therefore the angles CBE, EBD (10 *Def.*) are two right angles;

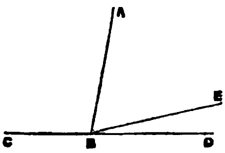
And because CBE is equal to the two angles CBA, ABE together, add the angle EBD to each of these equals; therefore the angles CBE, EBD are equal (2 *Ax.*) to the three angles CBA, ABE, EBD.

Again, because the angle DBA is equal to the two angles DBE, EBA, add to these equals the angle ABC, therefore the angles DBA, ABC are equal (2 *Ax.*) to the three angles DBE, EBA, ABC;

But the angles CBE, EBD have been demonstrated to be equal to the same three angles; and things that are equal to the same are equal (1 *Ax.*) to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC; but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore, when a straight line, etc. Q. E. D.

PROPOSITION XIV.

THEOR. *If, at a point in a straight line, two other straight lines, upon the opposite side of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.*



At the point B in the straight line AB, let the two straight lines BC, BD upon the opposite sides of AB, make the adjacent angles ABC, ABD equal together to two right angles, BD is in the same straight line with CB.

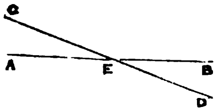
For, if BD be not in the same straight line with CB, let BE be in the same straight line with it; therefore, because the straight line AB makes angles with the straight line CBE, upon one side of it, the angles ABC, ABE are together equal (13. I.) to two right angles; but the angles ABC, ABD are likewise together equal (*Hyp.*) to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: take away the common angle ABC, the remaining angle ABE is equal (3 *Ax.*) to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And, in like manner, it may he demonstrated, that no other can be in the same straight line with it but BD, which therefore is in the same straight line with CB. Wherefore, if at a point, etc. Q. E. D.

PROPOSITION XV.

THEOR. *If two straight lines cut one another, the vertical or* opposite, *angles shall be equal.*

Let the two straight lines AB, CD cut one another in the point E; the angle AEC shall be equal to the angle DEB, and CEB to AED.

Because the straight line AE makes with CD the angles CEA, AED, these angles are together equal (13. I.) to two right angles. Again, because the straight line DE makes with AB the angles AED, DEB, these also are together equal (13. I.) to two right angles; and CEA, AED, have been demonstrated to be equal to two right angles; wherefore the angles CEA, AED, are equal (1 *Ax.*) to the angles AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal (3 *Ax.*) to the remaining angle DEB. In the same manner it can be demonstrated, that the angles CEB, AED are equal. Therefore, if two straight lines, etc. Q. E. D.



COR. 1. From this it is manifest, that, if two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

COR. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

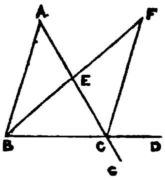
PROPOSITION XVI.

THEOR. *If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.*

Let ABC be a triangle, and let its side BC be produced to D, the exterior angle ACD is greater than either of the interior opposite angles CBA, BAC.

Bisect (10. I.) AC in E, join BE and produce it to F, and make EF equal (3. I.) to BE; join also FC, and produce AC to G.

Because AE is equal (*Constr.*) to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB, is equal (15. I.) to the angle CEF, because they are opposite vertical angles; therefore the base AB is equal (4. I.) to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF;



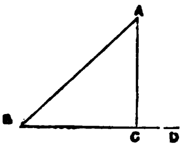
But the angle ECD is greater (9 *Ax.*) than the angle ECF; therefore the angle ACD is greater than BAE: in the same manner, if the side BC be bisected, it may be demonstrated that the angle BCG, that is (15. I.), the angle ACD, is greater than the angle ABC. Therefore, if one side, etc. Q. E. D.

PROPOSITION XVII.

THEOR. *Any two angles of a triangle are together less than two right angles.*

Let ABC be any triangle; any two of its angles together are less than two right angles.

Produce BC to D; and because ACD is the exterior angle of the triangle ABC, ACD is greater (16. I.) than the interior and opposite angle ABC; to each of these add the angle ACB; therefore the angles ACD, ACB are greater (4 *Ax.*) than the angles ABC, ACB; but ACD, ACB are together equal (13. I.) to two right angles; therefore the angles ABC, BCA are less than two right angles. In like manner, it may be demonstrated, that BAC, ACB, as also CAB, ABC, are less than two right angles. Therefore any two angles, etc. Q. E. D.

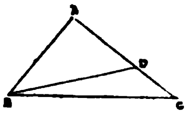


PROPOSITION XVIII.

THEOR. *The greater side of every triangle is opposite to the greater angle.*

Let ABC be a triangle, of which the side AC is greater than the side AB; the angle ABC is also greater than the angle BCA.

Because AC is greater than AB, make (3. I.) AD equal to AB, and join BD; and because ADB is the exterior angle of the triangle BDC, it is greater (16. I.) than the interior and opposite angle DCB; but ADB is equal (5. I.) to ABD, because the side AB is equal to the side AD: therefore the angle ABD is likewise greater than the angle ACB. Wherefore much more is the angle ABC greater than ACB. Therefore the greater side, etc. Q. E. D.

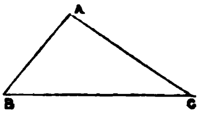


PROPOSITION XIX.

THEOR. *The greater angle of every triangle is subtended by the greater side, or* has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the side AC is likewise greater than the side AB.

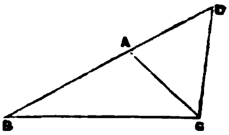
For, if it be not greater, AC must either be equal to AB or less than it; it is not equal, because then the angle ABC would be equal (5. I.) to the angle ACB; but it is not: therefore AC is not equal to AB; neither is it less; because then the angle ABC would be less (18. I.) than the angle ACB; but it is not; therefore the side AC is not less than AB; and it has been shown that it is not equal to AB; therefore AC is greater than AB. Wherefore the greater angle, etc. Q. E. D.



PROPOSITION XX.

THEOR. *Any two sides of a triangle are together greater than the third side.*

Let ABC be a triangle; any two sides of it together are greater than the third side, viz., the sides BA, AC greater than the side BC, and AB, BC, greater than AC, and BC, CA greater than AB.



Produce BA to the point D, and make (3. I.) AD equal to AC, and join DC.

Because DA is equal to AC, the angle ADC is likewise equal (5. I.) to ACD; but the angle BCD is greater (9 *Ax.*) than the angle ACD; therefore the angle BCD is greater than the angle ADC;

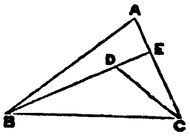
And because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater (19. I.) side is opposite to the greater angle, therefore the side DB is greater than the side BC: but DB is equal to BA and AC (*Constr.*), therefore the sides BA, AC are greater than BC.

In the same manner it may be demonstrated, that the sides AB, BC, are greater than CA, and BC, CA greater than AB. Therefore any two sides, etc. Q. E. D.

PROPOSITION XXI.

Theor. *If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

Let the two straight lines BD, CD be drawn from B, C, the ends of the side BC of the triangle ABC, to the point D within it; BD and DC are less than the other two sides BA, AC of the triangle, but contain an angle, BDC, greater than the angle BAC.



Produce BD to E; and because two sides of a triangle are greater than the third side, the two sides BA, AE, of the triangle ABE are greater than BE. To each of these add EC; therefore, the sides BA, AC are greater (4 *Ax.*) than BE, EC.

Again, because the two sides CE, ED of the triangle CED are greater than CD, add DB to each of these; therefore the sides CE, EB are greater than CD, DB;

But it has been shown that BA, AC are greater than BE, EC, much more then are BA, AC greater than BD, DC.

Again, because the exterior angle of a triangle (16. I.) is greater than the interior and opposite angle, the exterior angle BDC of the triangle CDE is greater than CED;

For the same reason, the exterior angle CEB of the triangle ABE is greater than BAC;

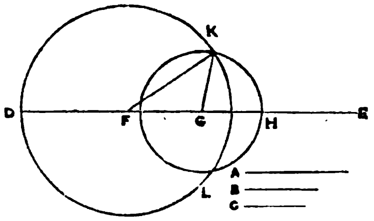
And it has been demonstrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, if from the ends of, etc. Q. E. D.

PROPOSITION XXII.

Prob. *To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third* (20. I.).

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third, viz., A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle, of which the *sides* shall be equal to A, B, C, each to each.

Take a straight line DE terminated at the point D, but unlimited towards E, and make (3. I) DF equal to A, FG to B, and GH equal to C; and from the centre F, at the distance FD, describe (3 *Post.*) the circle DKL; and from the centre G, at the distance GH, describe (3 *Post.*) another circle HLK, and join KF, KG; the triangle KFG has its sides equal to the three straight lines A, B, C.



Because the point F is the centre of the circle DKL, FD is equal (15 *Def.*) to FK; but FD is equal to the straight line A; therefore FK is equal (1 *Ax.*) to A.

Again, because G is the centre of the circle LKH, GH is equal (15 *Def.*) to GK; but GH is equal to C; therefore also GK is equal to C (1 *Ax.*):

And FG is equal to B; therefore the three straight lines KF, FG, GK, are equal to the three A, B, C; and, therefore, the triangle KFG has its three sides KF, FG, GK, equal to the three given straight lines A, B, C. Which was to be done.

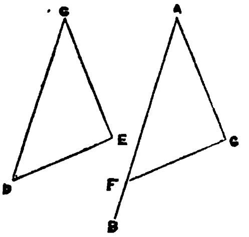
PROPOSITION XXIII.

PROB. *At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.*

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.

Take in CD, CE, any points D, E, and join DE, and make (22. I.) the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC, so that CD be equal to AF, CE to AG, and DE to FG;

And because DC, CE are equal to FA, AG, each to each, and the base DE to the base FG, the angle DCE is equal (8. I.) to the angle FAG. Therefore, at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.



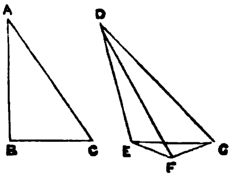
PROPOSITION XXIV.

Theor. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.*

Let ABC, DEF be two triangles, which have the two sides AB, AC, equal to the two DE, DF, each to each, viz., AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the base BC is also greater than the base EF.

Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in the straight line DE, make (23. I.) the angle EDG equal to the angle BAC; and make DG equal (3. I.) to AC or DF, and join EG, GF.

Because AB is equal (*Hyp.*) to DE, and AC (*Constr.*) to DG, the two sides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the base BC is equal (4. I.) to the base EG;



And because DG is equal to DF, the angle DFG is equal (5. I.) to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF (9 *Ax.*);

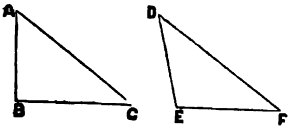
And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater (19. I.) side is opposite to the greater angle; the side EG is therefore greater than the side EF; but EG is equal to BC; and therefore also BC is greater than EF. Therefore, if two triangles, etc. Q. E. D.

PROPOSITION XXV.

Theor. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.*

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB equal to DE, and AC to DF; but the base CB greater than the base EF; the angle BAC is likewise greater than the angle EDF.

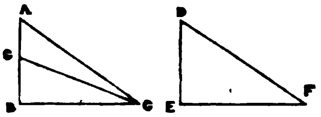
For, if it be not greater, it must be either equal to it, or less: but the angle BAC is not equal to the angle EDF; because then the base BC would be equal (4. I.) to EF; but it is not; therefore the angle BAC is not equal to the angle EDF;



Neither is it less; because then the base BC would be less (24. I.) than the base EF; but it is not; therefore the angle BAC is not less than the angle EDF; and it was shown that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, etc. Q. E. D.

PROPOSITION XXVI.

THEOR. *If two triangles have two angles of one equal to two angles of the other, each to each; and one side equal to one side, viz., either the sides adjacent to the equal angles, or the sides opposite to the equal angles in each; then shall the other sides be equal, each to each: and also the third angle of the one to the third angle of the other.*

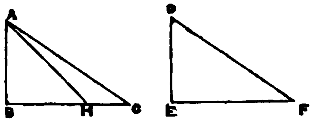


Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, each to each, viz., ABC to DEF, and BCA to EFD; also one side equal to one side; and first, let those sides be equal which are adjacent to the angles that are equal in the two triangles: viz., BC to EF; the other sides shall be equal, each to each, viz., AB to DE, and AC to DF, and the third angle BAC to the third angle EDF.

For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater of the two, and make BG (3. I.) equal to DE, and join GC; therefore, because BG is equal to DE, and BC to EF (*Hyp.*), the two sides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal (4. I.) to the base DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE;

But DFE is, by the hypothesis, equal to the angle BCA; wherefore also the angle BCG is equal (1 *Ax.*) to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE, that is, it is equal to it;

And BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal (*Hyp.*) to the angle DEF; the base, therefore, AC is equal (4. I.) to the base DF, and the third angle BAC to the third angle EDF.



Next let the sides which are opposite to equal angles in each triangle be equal to one another, viz., AB to DE; likewise in this case, the other sides shall be equal, AC to DF, and BC to EF; and also the third angle BAC to the third EDF.

For if BC be not equal to EF, let BC be the greater of them, and make BH equal (3. I.) to EF, and join AH;

And because BH is equal to EF, and AB to (*Hyp.*) DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal (*Hyp.*) angles; therefore the base AH is equal (4. I.) to the base DF, and the triangle ABH to the triangle DEF, and the other angles shall be equal, each to each, to which the equal sides are opposite; therefore the angle BHA is equal to the angle EFD;

But EFD is equal (*Hyp.*) to the angle BCA; therefore also the angle BHA is equal (1 *Ax.*) to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle BCA; which is impossible (16. I.); wherefore BC is not unequal to EF, that is, it is equal to it;

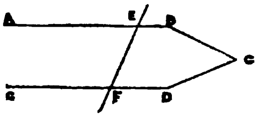
And AB is equal (*Hyp.*) to DE; therefore the two, AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the base AC is equal (4. I.) to the base DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, etc. Q. E. D.

PROPOSITION XXVII.

THEOR. *If a straight line, falling upon two other straight lines, makes the alternate angles equal to one another, these two straight lines shall be parallel.*

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD being produced shall meet either towards B, D, or towards A, C: let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater (16. I.) than the interior and opposite angle EFG; but it is also equal (*Hyp.*) to it, which is impossible; therefore AB and CD being produced do not meet towards B, D.

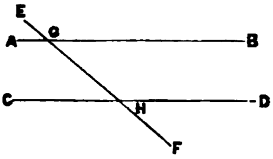


In like manner it may be demonstrated, that they do not meet towards A, C; but those straight lines which meet neither way, though produced ever so far, are parallel (35 *Def.*) to one another. AB therefore is parallel to CD. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION XXVIII.

THEOR. *If a straight line, falling upon two other straight lines, makes the exterior angle equal to the interior and opposite upon the same side of the line; or makes the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.*

Let the straight line EF, which falls upon the two straight lines AB, CD, make the exterior angle EGB equal to the interior and opposite angle GHD upon the same side; or make the interior angles on the same side BGH, GHD together equal to two right angles; AB is parallel to CD.



Because the angle EGB is equal (*Hyp.*) to the angle GHD, and the angle EGB equal (15. I.) to the angle AGH, the angle AGH is equal (1 *Ax.*) to the angle GHD; and they are the alternate angles; therefore AB is parallel (27. I.) to CD.

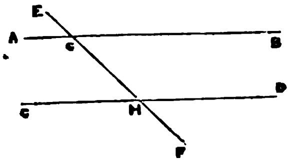
Again, because the angles BGH, GHD are equal (*Hyp.*) to two right angles; and that AGH, BGH are also equal (13. I.) to two right angles; the angles AGH, BGH are equal to the angles BGH, GHD: take away the common angle BGH: therefore the remaining angle

AGH is equal (3 *Ax.*) to the remaining angle GHD; and they are alternate angles; therefore AB is parallel (27. I.) to CD. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION XXIX.

Theor. *If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.*

Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles AGH, GHD, are equal to one another; and the exterior angle EGB is equal to the interior and opposite, upon the same side GHD; and the two interior angles BGH, GHD upon the same side, are together equal to two right angles.



For, if AGH be not equal to GHD, one of them must be greater than the other; let AGH be the greater; and because the angle AGH is greater than the angle GHD, add to each of them the angle BGH; therefore the angles AGH, BGH are greater (4 *Ax.*) than the angles BGH, GHD; but the angles AGH, BGH are equal (13. I.) to two right angles; therefore the angles BGH, GHD, are less than two right angles; but those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right angles, do meet (12 *Ax.*) together if continually produced; therefore the straight lines AB, CD, if produced far enough, shall meet; but they never meet, since they are parallel by the hypothesis; therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it;

But the angle AGH is equal (15. I.) to the angle EGB; therefore likewise EGB is equal (1 *Ax.*) to GHD; add to each of these the angle BGH; therefore the angles EGB, BGH are equal (2. *Ax.*) to the angles BGH, GHD; but EGB, BGH are equal (13. I.) to two right angles; therefore also BGH, GHD are equal (1 *Ax.*) to two right angles. Wherefore, if a straight line, etc. Q. E. D.

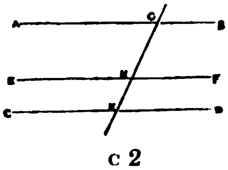
PROPOSITION XXX.

Theor. *Straight lines which are parallel to the same straight line are parallel to each other.*

Let AB, CD be each of them parallel to EF; AB is also parallel to CD.

Let the straight line GHK cut AB, EF, CD; and because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal (29. I.) to the angle GHF.

Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle GHF is equal (29. I.) to the angle GKD;



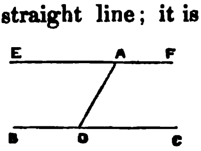
And it was shown that the angle AGH is

equal to the angle GHF; therefore also AGK is equal (1 *Ax.*) to GKD; and they are alternate angles; therefore AB is parallel (27. I.) to CD. Wherefore, straight lines, etc. Q. E. D.

PROPOSITION XXXI.

PROB. *To draw a straight line through a given point parallel to a given straight line.*

Let A be the given point, and BC the given straight line; it is required to draw a straight line through the point A, parallel to the straight line BC.



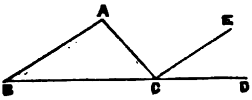
In BC take any point D, and join AD; and at the point A, in the straight line AD, make (23. I.) the angle DAE equal to the angle ADC; and produce the straight line EA to F.

Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel (27. I.) to BC. Therefore the straight line EAF is drawn through the given point A parallel to the given straight line BC. Which was to be done.

PROPOSITION XXXII.

THEOR. *If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.*

Let ABC be a triangle, and let one of its sides BC be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle, viz., ABC, BCA, CAB, are together equal to two right angles.



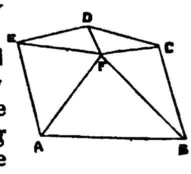
Through the point C draw CE parallel (31. I.) to the straight line AB; and because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE, are equal (29. I.)

Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal (29. I.) to the interior and opposite angle ABC;

But the angle ACE was shown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB, ABC;

To these equals add the angle ACB, and the angles ACD, ACB are equal (2 *Ax.*) to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal (13. I.) to two right angles: therefore also the angles CBA, BAC, ACB, are equal to two right angles. Wherefore, if a side of a triangle, etc. Q. E. D.

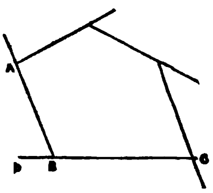
COR. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point F within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure; and the same angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is, (2 *Cor.* 15. I.) together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

Because every interior angle ABC, with its adjacent exterior ABD, is equal (13. I.) to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.



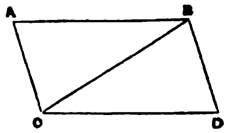
PROPOSITION XXXIII.

Theor. *The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.*

Let AB, CD be equal and parallel straight lines, and joined towards the same parts by the straight lines AC, BD; AC, BD are also equal and parallel.

Join BC; and because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (29. I.).

And because AB is equal (*Hyp.*) to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC, are equal to the two DC, CB: and the angle ABC is equal to the angle BCD; therefore the base AC is equal (4. I.) to the base BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles (4. I.), each to each, to which the equal sides are opposite; therefore the angle ACB is equal to the angle CBD;



And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel (27. I.) to BD; and it was shown to be equal to it. Therefore straight lines, etc. Q. E. D.

PROPOSITION XXXIV.

Theor. *The opposite sides and angles of a parallelogram are equal to one another, and the diameter* bisects *it, that is, divides it into two equal parts.*

N.B. A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.*

Let ACDB be a parallelogram, of which BC is a diameter; the opposite sides and angles of the figure are equal to one another; and the diameter BC bisects it.

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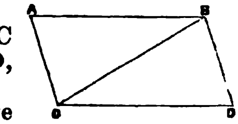
* In the case of any quadrilateral or four-sided figure which is not a parallelogram, the straight line which joins any two of its opposite angles is called a *diagonal.*

Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (29. I.) to one another;



And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD, are equal (29. I.) to one another;

Wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each, and one side BC common to the two triangles, which is adjacent to their equal angles; therefore their other sides shall be equal, each to each, and the third angle of the one to the third angle of the other (26. I.), viz., the side AB to the side CD, and AC to BD, and the angle BAC equal to the angle BDC:

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal (2 *Ax.*) to the whole angle ACD: and the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of parallelograms are equal to one another;

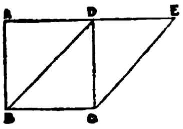
Also, their diameter bisects them; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal to the angle BCD; therefore the triangle ABC is equal (4. I.) to the triangle BCD, and the diameter BC divides the parallellogram ACDB into two equal parts. Q. E. D.

PROPOSITION XXXV.

THEOR. *Parallelograms upon the same base, and between the same parallels, are equal to one another.*

Let the parallelograms ABCD, EBCF be upon the same base BC, and between the same parallels AF; BC: the parallelogram ABCD shall be equal to the parallelogram EBCF.—*See* the 2nd and 3rd figures.

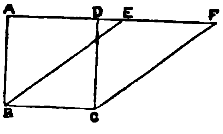
If the sides AD, EF of the parallelograms ABCD, EBCF, opposite to the base BC, be terminated in the same point D; it is plain that each of the parallelograms is double (34. I.) of the triangle BDC; and they are therefore equal (6 *Ax.*) to one another.

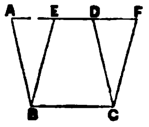


But, if the sides AD, EF, opposite to the base BC of the parallelograms ABCD, EBCF, be not terminated in the same point; then, because ABCD is a parallelogram, AD is equal (34. I.) to BC;

For the same reason EF is equal to BC; wherefore AD is equal (1. *Ax.*) to EF; and DE is common; therefore the whole, or the remainder AE, is equal (2 or 3 *Ax.*) to the whole, or the remainder DF:

AB also is equal to DC; and the two EA, AB are therefore equal to the two FD, DC, each to each; and the exterior angle FDC is equal (29. I.) to the interior EAB; therefore the base EB is equal to the base FC, and the triangle EAB equal (4. I.) to the triangle FDC.





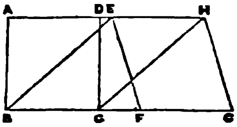
Take the triangle FDC from the trapezium ABCF, and from the same trapezium take the triangle EAB; the remainders therefore are equal (3 *Ax.*), that is, the parallelogram ABCD is equal to the parallelogram EBCF. Therefore parallelograms upon the same base, etc. Q. E. D.

PROPOSITION XXXVI.

THEOR. *Parallelograms upon equal bases, and between the same parallels, are equal to one another.*

Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG: the parallelogram ABCD is equal to EFGH.

Join BE, CH: and because BC is equal (*Hyp.*) to FG, and FG to (34. I.) EH, BC is equal to EH (1 *Ax.*); and they are parallels, and joined towards the same parts by the straight lines BE, CH. But straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel (33. I.); therefore EB, CH are both equal and parallel, and EBCH is a (*Def.* 34. I.) parallelogram; and it is equal (35. I.) to ABCD, because it is upon the same base BC, and between the same parallels BC, AD:



For the like reason, the parallelogram EFGH is equal to the same EBCH:

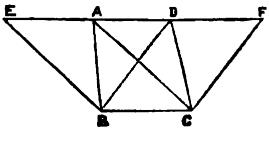
Therefore also the parallelogram ABCD is equal (1 *Ax.*) to EFGH. Wherefore parallelograms, etc. Q. E. D.

PROPOSITION XXXVII.

THEOR. *Triangles upon the same base, and between the same parallels, are equal to one another.*

Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC: the triangle ABC is equal to the triangle DBC.

Produce AD both ways to the points E, F, and through B draw (31. I.) BE parallel to CA; and through C draw CF parallel to BD; therefore each of the figures EBCA, DBCF is a (*Def.* 34. I.) parallelogram; and EBCA is equal (35. I.) to DBCF, because they are upon the same base BC, and between the same parallels BC, EF;



And the triangle ABC is the half of the parallelogram EBCA, because the diameter AB bisects (34. I.) it; and the triangle DBC is the half of the parallelogram DBCF, because the diameter DC bisects it:

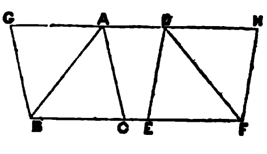
But the halves of equal things are equal (7 *Ax.*); therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles etc. Q. E. D.

PROPOSITION XXXVIII.

THEOR. *Triangles upon equal bases, and between the same parallels, are equal to one another.*

Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels BF, AD: the triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and through B draw BG parallel (31. I.) to CA, and through F draw FH parallel to ED: then each of the figures GBCA, DEFH, is a (*Def.* 34. I.) parallelogram; and they are equal (36. I.) to one another, because they are upon equal bases BC, EF, and between the same parallels BF, GH;



And the triangle ABC is the half (34. I.) of the parallelogram GBCA, because the diameter AB bisects it; and the triangle DEF is the half of the parallelogram DEFH, because the diameter DF bisects it:

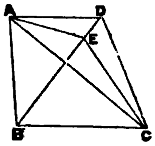
But the halves of equal things are equal (7 *Ax.*); therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles etc. Q. E. D.

PROPOSITION XXXIX.

THEOR. *Equal triangles upon the same base, and upon the same side of it, are between the same parallels.*

Let the equal triangles ABC, DBC be upon the same base BC, and upon the same side of it: they are between the same parallels.

Join AD; AD is parallel to BC; for, if it is not, through the point A draw (31. I.) AE parallel to BC, and join EC: the triangle ABC is equal (37. I.) to the triangle EBC, because it is upon the same base BC, and between the same parallels BC, AE:

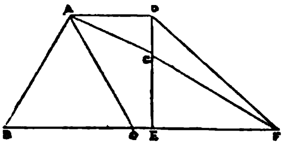


But the triangle ABC is equal (*Hyp.*) to the triangle BDC, therefore also the triangle BDC is equal (1 *Ax.*) to the triangle EBC, the greater to the less, which is impossible: therefore AE is not parallel to BC. In the same manner it can be demonstrated, that no other line but AD is parallel to BC; AD is therefore parallel to it. Wherefore equal triangles upon, etc. Q. E. D.

PROPOSITION XL.

THEOR. *Equal triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.*

Let the equal triangles ABC, DEF be upon equal bases BC, EF, in the same straight line BF, and towards the same parts; they are between the same parallels.



Join AD; AD is parallel to BC: for, if it is not, through A draw (31. I.) AG parallel to BF, and join GF: the triangle ABC is equal (38. I.) to the triangle GEF, because they are upon equal bases BC, EF, and between the same parallels BF, AG:

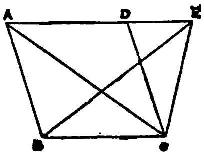
But the triangle ABC is equal (*Hyp.*) to the triangle DEF; therefore also the triangle DEF is equal (1 *Ax.*) to the triangle GEF, the greater to the less, which is impossible: therefore AG is not parallel to BF:

And in the same manner it can be demonstrated that there is no other parallel to it but AD: AD is therefore parallel to BF. Wherefore equal triangles, etc. Q. E. D.

PROPOSITION XLI.

THEOR. *If a parallelogram and a triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.*

Let the paralellogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE; the parallelogram ABCD is double of the triangle EBC.



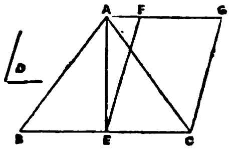
Join AC; then the triangle ABC is equal (37. I.) to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE.

But the parallelogram ABCD is double (34. I.) of the triangle ABC, because the diameter AC divides it into two equal parts; wherefore ABCD is also double (1 *Ax.*) of the triangle EBC. Therefore, if a parallelogram, etc. Q. E. D.

PROPOSITION XLII.

PROB. *To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let ABC be the given triangle, and D the given rectilineal angle. It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Bisect (10. I.) BC in E, join AE, and at the point E in the straight line EC make (23. I.) the angle CEF equal to D; and through A draw (31. I.) AG parallel to EC, and through C draw CG parallel to EF: therefore FECG is a (*Def.* 34. I.) parallelogram.

And because BE is equal (*Constr.*) to EC, the triangle ABE is likewise equal (38. I.) to the triangle AEC, since they are upon equal bases BE, EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle AEC.

And the parallelogram FECG is likewise double (41. I.) of the triangle AEC, because it is upon the same base, and between the same parallels:

Therefore the parallelogram FECG is equal (6 *Ax.*) to the triangle ABC, and it has one of its angles CEF equal to the given angle D; wherefore there has been described a parallelogram FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D. Which was to be done.

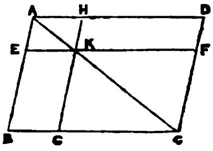
PROPOSITION XLIII.

Theor. *The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.*

Let ABCD be a parallelogram, of which the diameter is AC, and EH, FG the parallelograms about AC, *that is, through which* AC *passes*, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called the complements. The complement BK is equal to the complement KD.

Because ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal (34. I.) to the triangle ADC: and because EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal to the triangle AHK:

For the same reason, the triangle KGC is equal to the triangle KFC:



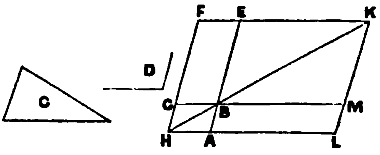
Then, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC, is equal (2 *Ax.*) to the triangle AHK together with the triangle KFC: but, the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal (3 *Ax.*) to the remaining complement KD. Wherefore the complements, etc. Q. E. D.

PROPOSITION XLIV.

Prob. *To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.*

Let AB be the given straight line, C the given triangle, and D the given rectilineal angle. It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make (42. I.) the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, so that BE be in the same straight line with AB, and produce FG to H, and through A draw (31. I.) AH parallel to BG or EF, and join HB.



Then, because the straight line HF falls upon the parallels AH, EF, the angles AHF, HFE are together equal (29. I.) to two right angles; wherefore the angles BHF, HFE are less than two right angles; but straight lines which, with another straight line, make the interior angles upon the same side less than two right angles, do meet (12 *Ax.*) if produced far enough; therefore HB, FE shall meet if produced;

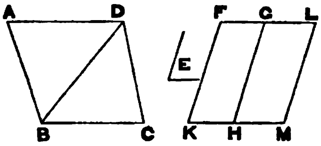
Let them meet in K, and through K draw KL parallel to EA or FH (31. I.) and produce HA, GB to the points L, M: then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK; and LB, BF are the complements; therefore LB is equal (43. I.) to BF:

But BF is equal to the triangle C; wherefore LB is equal (1 *Ax.*) to the triangle C; and because the angle GBE is equal (15. I.) to the angle ABM, and likewise (*Constr.*) to the angle D; the angle ABM is equal (1 *Ax.*) to the angle D. Therefore the parallelogram LB is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Which was to be done.

PROPOSITION XLV.

Prob. ***To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.***

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.



Join DB, and describe (42. I.) the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E; and to the straight line GH apply (44. I.) the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E; and because the angle E is equal (*Constr.*) to each of the angles FKH, GHM, the angle FKH is equal (1 *Ax.*) to GHM: add to each of these the angle KHG; therefore the angles FKH, KHG are equal (2 *Ax.*) to the angles KHG, GHM; but FKH, KHG are equal (29. I.) to two right angles; therefore also KHG, GHM are equal (1 *Ax.*) to two right angles: and because at the point H in the straight line GH, the two straight lines KH, HM upon the opposite sides of it make the adjacent angles equal to two right angles, KH is in the same straight line (14. I.) with HM:

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF (29. I.) are equal: add to each of these the angle HGL; therefore the angles MHG, HGL are equal (2 *Ax.*) to the angles HGF, HGL: but the angles MHG, HGL are equal (29. I.) to two right angles; wherefore also the angles HGF, HGL are equal (1 *Ax.*) to two right angles, and FG is therefore in the same straight line (14. I.) with GL;

And because KF is parallel to HG (*Constr.*), and HG to ML; KF is parallel (30. I.) to ML; and KM, FL are parallels; wherefore KFLM is (*Def.* 34. I.) a parallelogram; and because the triangle ABD is equal to the parallelogram HF (*Constr.*), and the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM (2 *Ax.*); therefore the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.

Cor. From this it is manifest how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, viz., by applying (44. I.) to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle.

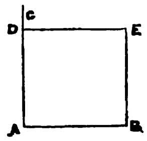
PROPOSITION XLVI.

Prob. *To describe a square upon a given straight line.*

Let AB be the given straight line; it is required to describe a square upon AB.

From the point A draw (11. I.) AC at right angles to AB; and make (3. I.) AD equal to AB, and through the point D draw DE parallel (31. I.) to AB, and through B draw BE parallel to AD; therefore ADEB is a (*Def.* 34. I.) parallelogram: whence AB is equal (34. I.) to DE, and AD to BE; but BA is equal to AD; therefore the four straight lines BA, AD, DE, EB are equal (1 *Ax.*) to one another, and the parallelogram ADEB is equilateral:

Likewise all its angles are right angles; because the straight line AD meeting the parallels AB, DE, the angles BAD, ADE are equal (29. I.) to two right angles: but BAD is a (*Constr.*) right angle; therefore also ADE is a right angle; but the opposite angles of parallelograms are equal (34. I.); therefore each of the opposite angles ABE, BED is a (1 *Ax.*) right angle; wherefore the figure ADEB is rectangular; and it has been demonstrated that it is equilateral; it is therefore a (30 *Def.*) square, and it is described upon the given straight line AB. Which was to be done.



Cor. Hence every parallelogram that has one right angle has all its angles right angles.

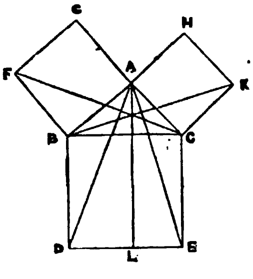
PROPOSITION XLVII.

Theor. *In any right-angled triangle, the square which is described upon the side subtending the right angle is equal to the squares described upon the sides which contain the right angle.*

Let ABC be a right-angled triangle having the right angle BAC; the square described upon the side BC is equal to the squares described upon BA, AC.

On BC describe (46. I.) the square BDEC, and on BA, AC the squares GB, HC; and through A draw (31. I.) AL parallel to BD or CE, and join AD, FC. Then, because each of the angles BAC (*Hyp.*), BAG is a right angle (30 *Def.*) the two straight lines AC, AG, upon the opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the same straight line (14. I.) with AG;

For the same reason, AB and AH are in the same straight line;



And because the angle DBC is equal (11 *Ax.*) to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is equal (2 *Ax.*) to the whole FBC ; and because the two sides AB, BD are equal (30 *Def.*) to the two FB, BC, each to each, and the angle DBA equal to the angle FBC ; therefore the base AD is equal (4. I.) to the base FC, and the triangle ABD to the triangle FBC :

Now the parallelogram BL is double (41. I.) of the triangle ABD, because they are upon the same base BD, and between the same parallels BD, AL ; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC. But the doubles of equals are equal (6 *Ax.*) to one another ; therefore the parallelogram BL is equal to the square GB :

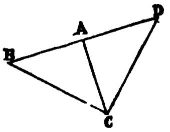
And, in the same manner, by joining AE, BK, it is demonstrated, that the parallelogram CL is equal to the square HC ;

Therefore the whole square BDEC is equal (2 *Ax.*) to the two squares GB, HC ; and the square BDEC is described upon the straight line BC, and the squares GB, HC upon BA, AC ; wherefore the square upon the side BC is equal to the squares upon the sides BA, AC. Therefore, in any right-angled triangle, etc. Q. E. D.

PROPOSITION XLVIII.

THEOR. *If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.*

If the square described upon BC, one of the sides of the triangle ABC, be equal to the squares upon the other sides BA, AC, the angle BAC is a right angle.



From the point A draw (11. I.) AD at right angles to AC, and make (3. I.) AD equal to BA, and join DC : then, because DA is equal to AB, the square of DA is equal to the square of AB ; to each of these add the square of AC ; therefore the squares of DA, AC are equal (2 *Ax.*) to the squares of BA, AC :

But the square of DC is equal (47. I.) to the squares of DA, AC, because DAC is a (*Constr.*) right angle ; and the square of BC, by hypothesis, is equal to the squares of BA AC ; therefore the square of DC is equal (1 *Ax.*) to the square of BC ; and therefore also the side DC is equal to the side BC.

And because the side DA is equal (*Constr.*) to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC ; and the base DC is equal to the base BC ; therefore the angle DAC is equal (8. I.) to the angle BAC : but DAC is a right angle ; therefore also BAC is a right angle. Therefore, if the square, etc. Q. E. D.

EXERCISES ON BOOK I.

1. If a straight line meet another so as to make two adjacent angles, and each of these angles be bisected, the bisecting lines will be at right angles to one another.

2. If through the vertex of an isosceles triangle a line be drawn parallel to the base, it will bisect the angles at the vertex made by producing the equal sides of the triangle.

3. The difference between any two sides of a triangle is less than the third side.

4. If two straight lines which meet one another be parallel to two other lines which meet one another, the angle contained by the first two will be equal to the angle contained by the other two.

5. Vary the second part of the demonstration of i. 32 so as to be effected without producing either side of the triangle.

6. Prove the second corollary of i. 32 independently of the first, and from it deduce the first. Also,

   (*a*) Show that each angle of an equilateral triangle is two-thirds of a right angle;

   (*b*) That the two acute angles of a right-angled triangle are equal to one right angle;

   (*c*) If a triangle have its exterior angle and likewise one of its interior angles *double* of those in another triangle, its remaining opposite interior angle is *double* of the corresponding angle in the other.*

7. The diagonals of a parallelogram bisect one another at their point of intersection, and any line drawn through their intersection bisects the parallelogram.

8. If through the middle of a line intercepted between two parallel lines, any other line be drawn terminating in those parallels, then this line will be bisected at that point.

9. (*a*) If one of the sides of a triangle be divided into any number of equal parts, and lines be drawn through the several points of division parallel to the base, these parallels will divide the other side into the same number of equal parts. Also (*b*), the second, third, fourth, etc. of the parallels intercepted between the sides, will be twice, three times, four times, etc. the first intercepted parallel.† (*c*) And if the sides of the triangle be produced either below the base or above the vertex, and parts be taken in these prolongations equal to the former parts, the same relations as in (*a*) and (*b*) will hold good.

10. Divide a given line into any given number of equal parts.

11. (*a*) If two sides of a triangle be bisected, the line which joins the points of bisection will be parallel to the base, and equal to half the base. (*b*) If lines be drawn from the points of bisection to the opposite angles, they will all meet in one point; and each will be divided at that point into segments such that those drawn to the angles are respectively the doubles of those drawn to the bisecting points.

12. If the three sides of a triangle be bisected and the points of

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* This is the foundation of the construction of Hadley's sextant.

† The construction of the *diagonal scale* depends upon this property.

bisection two and two be joined, four triangles will be formed equiangular with the first, and each equal to one fourth of that triangle.

13. If one angle of a parallelogram be a right angle, all the angles are right angles: and if likewise any two adjacent sides be equal, it is a square. Prove also that if all the sides of a quadrilateral figure be bisected, and lines be drawn to join the adjacent points of bisection, they will form a parallelogram equal to half the given quadrilateral.

14. If the two diagonals of a parallelogram be equal, all the angles of the parallelogram are right angles.

15. If the four sides of a quadrilateral figure be equal, the figure is a parallelogram, and the diagonals are perpendicular to one another.

16. If the right angle of a right-angled triangle be bisected by a line which also cuts the hypothenuse, and from the point of intersection lines be drawn parallel to the sides, a square will be formed of which the diagonal is the bisecting line.

17. Equal squares have equal sides, and the squares described upon equal lines are equal.

18. In the figure to Prop. 47, if BC be produced both ways, and perpendiculars be drawn to it from F and K, then

(*a*) These perpendiculars will be respectively equal to the segments of BC made by AL;

(*b*) They will meet BC at distances from B and C respectively, each equal to the perpendicular from A upon BC;

(*c*) The lines BK, CF are respectively perpendicular to AE, AD;

(*d*) The squares described on the sides BA, AC about the right angle are equal to the rectangles under the hypothenuse and the segments of it made by the perpendicular from A upon BC respectively adjacent to BA, AC.

19. Describe a square equal to any number of given squares, and likewise a square equal to the difference between two given squares.

20. In four several triangles there are alike given the base and one of the angles at the base; and the remaining data in the several cases are:—

(*a*) The perpendicular;
(*b*) The sum of the sides;
(*c*) The difference of the sides;
(*d*) The line drawn from the middle of the base to the vertex.

It is required to construct these triangles.

21. (*α*) Given two straight lines, to exhibit half their sum and half their difference by a construction.

(*β*) Given two rectilinear angles to construct angles equal to half their sum and half their difference.

22. If an angle of a triangle be two-thirds of a right angle, and the sides containing it equal, the triangle is equilateral.

23. If one acute angle of a right-angled triangle be equal to one of another, the triangles will be equiangular. Will they be equal?

24. If two angles of a triangle be together equal to one right angle, the triangle will be right-angled; if less, obtuse-angled; and if greater, acute-angled.

25. Let two right-angled triangles have any one of the three sides of the first equal to a corresponding one of the second; and any one

of the remaining parts of the first greater than the corresponding part of the other: it is required to compare (as to greater and less) all the other parts of one triangle with the corresponding parts of the other.

26. From the right angle of a triangle draw a line to meet the hypothenuse, making an angle with one side equal to the adjacent acute angle of the triangle: then

(*a*) This line will make with the other side an angle equal to the other acute angle of the triangle;

(*b*) It will make with the perpendicular from the right angle upon the hypothenuse, an angle equal to the difference of the acute angles of the triangle.

(*c*) It will itself be equal to each of the segments into which it divides the hypothenuse;

27. Only one perpendicular can be drawn from a given point to a given straight line, whether the point be in the line or without it.

28. The perpendicular is the shortest line that can be drawn to a line from a point without it.

29. Of straight lines drawn to a given straight line from a given point without it, that which is nearer to the perpendicular is less than the more remote; and those which are equally distant are equal.

30. Equal straight lines drawn from a given point to a given straight line, meet it at equal distances from the perpendicular; they also make equal angles with the perpendicular; and equal angles with the given straight line.

31. Given two sides and one of the angles of a triangle to construct the triangle.

32. Two right-angled triangles are *wholly* equal, when :—

(*a*) The two sides including the right angles are equal each to each;

(*b*) The hypothenuse and one side are equal each to each;

(*c*) The hypothenuse and either acute angle are equal each to each;

(*d*) An acute angle and the side opposite are equal, each to each;

(*e*) An acute angle and the side adjacent are equal, each to each.

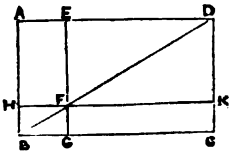
33. If two lines be perpendicular to two others which meet one another, these will intersect and contain an angle equal to that of the first two lines.

BOOK II.

DEFINITIONS.

1. Every right-angled parallelogram is said to be contained by any two of the straight lines which contain one of the right angles.

2. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. "Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon."



PROPOSITIONS.

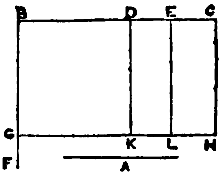
PROPOSITION I.

Theor. *If there be two straight lines, one of which is divided into any number of parts; the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.*

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the straight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

From the point B draw (11. i.) BF at right angles to BC, and make BG equal (3. i.) to A; and through G draw (31. i.) GH parallel to BC; and through D, E, C, draw (31. i.) DK, EL, CH parallel to BG;

Then the rectangle BH is equal to the rectangles BK, DL, EH;



And BH is contained by A, BC, for it is contained by GB, BC, and GB is equal (*Constr.*) to A;

And BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A;

And DL is contained by A, DE, because DK, that is (34. i.) BG, is equal to A;

And in like manner the rectangle EH is contained by A, EC:

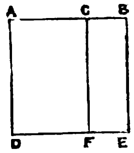
Therefore the rectangle contained by A, BC, is equal to the several rectangles contained by A, BD, and by A, DE; and also by A, EC. Wherefore, if there be two straight lines, etc. Q. E. D.

PROPOSITION II.

Theor. *If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.*

Let the straight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle* AB, AC, shall be equal to the square of AB.

Upon AB describe (46. I.) the square ADEB, and through C draw (31. I.) CF, parallel to AD or BE:



Then AE is equal to the rectangles AF, CE;

And AE is the square of AB;

And AF is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB;

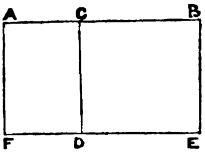
Therefore the rectangle contained by AB, AC, together with the rectangle AB, BC, is equal to the square of AB. If therefore a straight line, etc. Q. E. D.

PROPOSITION III.

Theor. *If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.*

Let the straight line AB be divided into any two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC.

Upon BC describe (46. I.) the square CDEB, and produce ED to F, and through A draw (31. I.) AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE;



And AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC;

And AD is contained by AC, CB, for CD is equal to CB;

And DB is the square of BC;

Therefore the rectangle AB, BC is equal to the rectangle AC, CB, together with the square of BC. If therefore a straight line, etc. Q. E. D.

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* N.B.—To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines AB, AC is sometimes simply called the rectangle AB, AC.

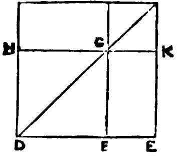
PROPOSITION IV.

Theor. *If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.*

Let the straight line AB be divided into any two parts in C; the square of AB is equal to the squares of AC, CB, and to twice the rectangle contained by AC, CB.

Upon AB describe (46. I.) the square ADEB, and join BD, and, through C draw (31. I.) CGF parallel to AD or BE, and through G draw HK parallel to AB or DE:

And because CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal (29. I.) to the interior and opposite angle ADB; but ADB is equal (5. I.) to the angle ABD, because BA is equal (30 *Def.*) to AD, being sides of a square; wherefore the angle CGB is equal (1 *Ax.*) to the angle GBC; and therefore the side BC is equal (6. I.) to the side CG: but CB is equal (34. I.) also to GK, and CG to BK; wherefore the figure CGKB is equilateral:



It is likewise rectangular; for CG is parallel to BK, and CB meets them; the angles KBC, GCB are therefore equal to two right angles; and KBC is a right angle; wherefore GCB is a right angle: and therefore also the angles (34. I.) CGK, GKB opposite to these, are right angles, and CGKB is rectangular; but it is also equilateral, as was demonstrated; wherefore it is a square, and it is upon the side CB:

For the same reason HF also is a square, and it is upon the side HG, which is equal (34. I.) to AC: therefore HF, CK are the squares of AC, CB:

And because the complement AG is equal (43. I.) to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB: therefore GE is also equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB:

And HF, CK are the squares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the squares of AC, CB, and to twice the rectangle AC, CB:

But HF, CK, AG, GE make up the whole figure ADEB, which is the square of AB: therefore the square of AB is equal to the squares of AC, CB, and twice the rectangle AC, CB. Wherefore, if a straight line, etc. Q. E. D.

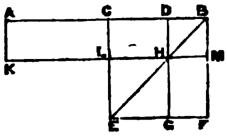
Cor. From the demonstration, it is manifest, that the parallelograms about the diameter of a square are likewise squares.

PROPOSITION V.

Theor. *If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.*

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the square of CD, is equal to the square of CB.

Upon CB describe (46. I.) the square CEFB, join BE, and through D draw (31. I.) DHG parallel to CE or BF; and through H draw KLM parallel to CB or EF; and also through A draw AK parallel to CL or BM:



And because the complement CH is equal (43. I.) to the complement HF, to each of these add DM; therefore the whole CM is equal (2 *Ax.*) to the whole DF; but CM is equal (36. I.) to AL, because AC is equal to CB; therefore also AL is equal to DF.

To each of these add CH, and the whole AH is equal to DF and CH:

But AH is the rectangle contained by AD, DB, for DH is equal (4. *Cor.* II.) to DB; and DF together with CH is the gnomon CMG;

Therefore the gnomon CMG is equal to the rectangle AD, DB:

To each of these add LG, which is equal (4. *Cor.* II.) to the square of CD:

Therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square of CD;

But the gnomon CMG and LG make up the whole figure CEFB, which is the square of CB: therefore the rectangle AD, DB, together with the square of CD is equal to the square of CB. Wherefore, if a straight line, etc. Q. E. D.

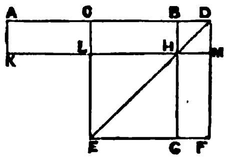
From this proposition it is manifest, that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum and difference.

PROPOSITION VI.

THEOR. *If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.*

Let the straight line AB be bisected in C, and produced to the point D; the rectangle AD, DB, together with the square of CB, is equal to the square of CD.

Upon CD describe (46. I.) the square CEFD, join DE, and through B draw (31. I.) BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and also through A draw AK parallel to CL or DM;



And because AC is equal to CB, the rectangle AL is equal (36. I.) to CH; but CH is equal (43. I.) to HF; therefore also AL is equal to HF:

To each of these add CM; therefore the whole AM is equal to the gnomon CMG:

And AM is the rectangle contained by AD, DB for DM is equal (4. *Cor.* II.) to DB: therefore the gnomon CMG is equal to the rectangle AD, DB:

Add to each of these LG, which is equal to the square of CB;

therefore the rectangle AD, DB, together with the square of CB, is equal to the gnomon CMG, and the figure LG;

But the gnomon CMG and LG make up the whole figure CEFD, which is the square of CD;

Therefore the rectangle AD, DB, together with the square of CB, is equal to the square of CD. Wherefore if a straight line, &c. Q.E.D.

PROPOSITION VII.

THEOR. *If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.*

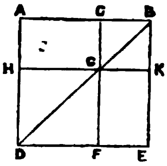
Let the straight line AB be divided into any two parts in the point C; the squares of AB, BC are equal to twice the rectangle AB, BC, together with the square of AC.

Upon AB describe (46. I.) the square ADEB, and construct the figure as in the preceding propositions;

And because AG is equal (43. I.) to GE, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE are double of AK:

But AK, CE are the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is double of AK:

But twice the rectangle AB, BC is double of AK, for BK is equal (4. *Cor.* II.) to BC; therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC.



To each of these equals add HF, which is equal to the square of AC; therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB, BC, and the square of AC:

But the gnomon AKF, together with the squares CK, HF, make up the whole figure ADEB and CK, which are the squares of AB and BC: therefore the squares of AB and BC are equal to twice the rectangle AB, BC, together with the square of AC. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION VIII.

THEOR. *If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line, which is made of the whole and that part.*

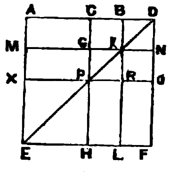
Let the straight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the square of AC, is equal to the square of the straight line made up of AB and BC together.

Produce AB to D, so that BD be equal to CB, and upon AD describe the square AEFD; and construct two figures such as in the preceding.

Because CB is equal to BD, and that CB is equal (34. I.) to GK, and BD to KN; therefore GK is equal to KN:

For the same reason, PR is equal to RO;

And because CB is equal to BD, and GK to KN, the rectangle CK is equal (36. i.) to BN, and GR to RN; but CK is equal (43. i.) to RN, because they are the complements of the parallelogram CO; therefore also BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another, and so are quadruple of one of them CK:



Again, because CB is equal to BD, and that BD is equal (4. Cor. ii.) to BK, that is, to CG,, and CB equal to GK, that is, to GP; therefore CG is equal to GP:

And because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF:

But MP is equal (43. i.) to PL because they are the complements of the parallelogram ML: wherefore AG is equal also to RF: therefore the four rectangles AG, MP, PL, RF, are equal to one another, and so are quadruple of one of them AG.

And it was demonstrated that the four CK, BN, GR, and RN are quadruple of CK. Therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK;

And because AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC is quadruple of AK: but the gnomon AOH was demonstrated to be quadruple of AK; therefore four times the rectangle AB, BC, is equal to the gnomon AOH.

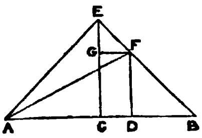
To each of these add XH, which is equal (4. Cor. ii.) to the square of AC: therefore four times the rectangle AB, BC, together with the square of AC, is equal to the gnomon AOH and the square XH:

But the gnomon AOH and XH make up the figure AEFD, which is the square of AD: therefore four times the rectangle AB, BC, together with the square of AC, is equal to the square of AD, that is of AB and BC added together in one straight line. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION IX.

Theor. *If a straight line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.*

Let the straight line AB be divided at the point C into two equal, and at D into two unequal parts; the squares of AD, DB are together double of the squares of AC, CD.



From the point C draw (11. i.) CE at right angles to AB, and make it equal to AC or CB, and join EA, EB; through D draw (31. i.) DF parallel to CE, and through F draw FG parallel to AB; and join AF:

Then because AC is equal to CE, the angle EAC is equal (5. i.) to the angle AEC; and because the angle ACE is a right angle, the two others AEC, EAC together make one right angle (32. i.); and they are equal to one another; each of them therefore is half of a right angle.

For the same reason each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle:

And because the angle GEF is half a right angle, and EGF a right angle, for it is equal (29. I.) to the interior and opposite angle ECB, the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the side EG equal (6. I.) to the side GF.

Again, because the angle at B is half a right angle, and FDB a right angle, for it is equal (29. I.) to the interior and opposite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the side DF to (6. I.) the side DB.

And because AC is equal to CE, the square of AC is equal to the square of CE; therefore the squares of AC, CE are double of the square of AC:

But the square of EA is equal (47. I.) to the squares of AC, CE, because ACE is a right angle; therefore the square of EA is double of the square of AC:

Again, because EG is equal to GF, the square of EG is equal to the square of GF; therefore the squares of EG, GF are double of the square of GF:

But the square of EF is equal to the squares of EG, GF; therefore the square of EF is double of the square of GF; and GF is equal (34. I.) to CD; therefore the square of EF is double of the square of CD:

But the square of AE is likewise double of the square of AC; therefore the squares of AE, EF are double of the squares of AC, CD: and the square of AF is equal (47. I.) to the squares of AE, EF, because AEF is a right angle; therefore the square of AF is double of the squares of AC, CD: but the squares of AD, DF are equal to the square of AF, because the angle ADF is a right angle; therefore the squares of AD, DF are double of the squares of AC, CD: and DF is equal to DB; therefore the squares of AD, DB are double of the squares of AC, CD. If therefore a straight line, etc. Q. E. D.

PROPOSITION X.

THEOR. *If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.*

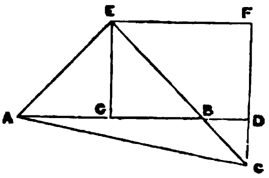
Let the straight line AB be bisected in C, and produced to the point D: the squares of AD, DB are double of the squares of AC, CD.

From the point C draw (11. I.) CE at right angles to AB; and make it equal to AC or CB, and join AE, EB; through E draw (31. I.) EF parallel to AB, and through D draw DF parallel to CE. And because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal (29. I.) to two right angles; and therefore the angles BEF, EFD are less than two right angles: but straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet (12 *Ax.*) if produced far enough; therefore EB, FD shall meet, if produced, towards B, D: let them meet in G, and join AG.

Then, because AC is equal to CE, the angle CEA is equal (5. I.) to the angle EAC; and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half a right angle. (32. I.):

For the same reason, each of the angles CEB, EBC is half a right angle; therefore AEB is a right angle:

And because EBC is half a right angle, DBG is also (15. I.) half a right angle, for they are vertically opposite: but BDG is a right angle, because it is equal (29. I.) to the alternate angle DCE; therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG; wherefore also the side BD is equal (6. I.) to the side DG.



Again, because EGF is half a right angle, and that the angle at F is a right angle, because it is equal (34. I.) to the opposite angle ECD, the remaining angle FEG is half a right angle, and equal to the angle EGF; wherefore also the side GF is equal (6. I.) to the side FE.

And because EC is equal to CA, the square of EC is equal to the square of CA; therefore the squares of EC, CA are double of the square of CA: but the square of EA is equal (47. I.) to the squares of EC, CA; therefore the square of EA is double of the square of AC:

Again, because GF is equal to FE, the square of GF is equal to the square of FE; and therefore the squares of GF, FE are double of the square of EF; but the square of EG is equal (47. I.) to the squares of GF, FE; therefore the square of EG is double of the square of EF; and EF is equal to CD; wherefore the square of EG is double of the square of CD.

But it was demonstrated, that the square of EA is double of the square of AC; therefore the squares of AE, EG are double of the squares of AC, CD:

And the square of AG is equal (47. I.) to the squares of AE, EG; therefore the square of AG is double of the squares of AC, CD:

But the squares of AD, GD are equal (47. I.) to the square of AG; therefore the squares of AD, DG are double of the squares of AC, CD:

But DG is equal to DB; therefore the squares of AD, DB are double of the squares of AC, CD. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION XI.

PROB. *To divide a given straight line into two parts, so that the rectangle contained by the whole, and one of the parts, shall be equal to the square of the other part.*

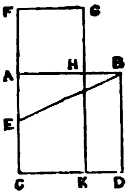
Let AB be the given straight line; it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Upon AB describe (46. I.) the square ABDC; bisect (10. I.) AC in E, and join BE; produce CA to F, and make (3. I.) EF equal to EB; and upon A F describe (46. I.) the square FGHA: AB is divided in H, so that the rectangle AB, BH is equal to the square of AH.

Produce GH to K:

Because the straight line AC is bisected in E, and produced to the point F, the rectangle CF, FA, together with the square of AE, is equal (6. II.) to the square of EF:

But EF is equal to EB; therefore the rectangle CF, FA, together with the square of AE, is equal to the square of EB:



And the squares of BA, AE are equal (47. I.) to the square of EB, because the angle EAB is a right angle; therefore the rectangle CF, FA, together with the square of AE, is equal to the squares of BA, AE:

Take away the square of AE which is common to both; therefore the remaining rectangle CF, FA is equal (3 *Ax.*) to the square of AB:

And the figure FK is the rectangle contained by CF, FA, for FA is equal (30 *Def.*) to FG; and AD is the square of AB, therefore FK is equal (1 *Ax.*) to AD:

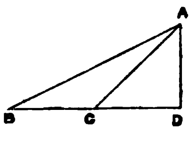
Take away the common part AK, and the remainder FH is equal (3 *Ax.*) to the remainder HD: and HD is the rectangle contained by AB, BH, for AB is equal (30 *Def.*) to BD; and FH is the square of AH. Therefore the rectangle AB, BH is equal to the square of AH: wherefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square of AH. Which was to be done.

PROPOSITION XII.

THEOR. *In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.*

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB, and from the point A let AD be drawn (12. I.) perpendicular to BC produced: the square of AB is greater than the squares of AC, CB, by twice the rectangle BC, CD.

Because the straight line BD is divided into two parts in the point C, the square of BD is equal (4. II.) to the squares of BC, CD, and twice the rectangle BC, CD:



To each of these equals add the square of DA; and the squares of BD, DA are equal to the squares of BC, CD, DA, and twice the rectangle BC, CD:

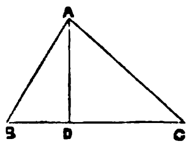
But the square of BA is equal (47. I.) to the squares of BD, DA, because the angle at D is a right angle; and the square of CA is equal (47. I.) to the squares of CD, DA;

Therefore the square of BA is equal to the squares of BC, CA, and twice the rectangle BC, CD; that is, the square of BA is greater than the squares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuse-angled triangles, etc. Q. E. D.

PROPOSITION XIII.

Theor. *In every triangle, the square of the side subtending any of the acute angles is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.*

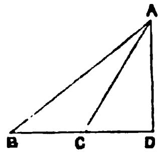
Let ABC be any triangle, and the angle at B one of its acute angles; and upon BC, one of the sides containing it, let fall the perpendicular (12. I.) AD from the opposite angle: the square of AC, opposite to the angle B, is less than the squares of CB, BA, by twice the rectangle CB, BD.



First, let AD fall within the triangle ABC: and because the straight line CB is divided into two parts in the point D, the squares of CB, BD are equal (7. II.) to twice the rectangle contained by CB, BD, and the square of DC:

To each of these equals add the square of AD; therefore the squares of CB, BD, DA are equal (2 *Ax.*) to twice the rectangle CB, BD, and the squares of AD, DC:

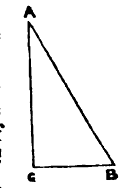
But the square of AB is equal (47. I.) to the squares of BD, DA, because the angle BDA is a right angle; and the square of AC is equal to the squares of AD, DC; therefore the squares of CB, BA are equal to the square of AC, and twice the rectangle CB, BD; that is, the square of AC alone is less than the squares of CB, BA, by twice the rectangle CB, BD.



Secondly, let AD fall without the triangle ABC: then, because the angle at D is a right angle, the angle ACB is greater (16. I.) than a right angle; and therefore the square of AB is equal (12. II.) to the squares of AC, CB, and twice the rectangle BC, CD:

To these equals add the square of BC, and the squares of AB, BC are equal (2 *Ax.*) to the square of AC, and twice the square of BC, and twice the rectangle BC, CD:

But because BD is divided into two parts in C, the rectangle DB, BC is equal (3. II.) to the rectangle BC, CD and the square of BC; and the doubles of these are equal: therefore the squares of AB, BC are equal to the square of AC, and twice the rectangle DB, BC: therefore the square of AC alone is less than the squares of AB, BC, by twice the rectangle DB, BC.



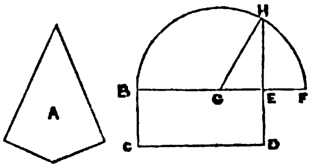
Lastly, let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B: and it is manifest, that the squares of AB, BC, are equal (47. I. *and* 2 *Ax.*) to the square of AC and twice the square of BC. Therefore, in every triangle, etc. Q. E. D.

PROPOSITION XIV.

PROB. *To describe a square that shall be equal to a given rectilineal figure.*

Let A be the given rectilineal figure: it is required to describe a square that shall be equal to A.

Describe (45. I.) a rectangular parallelogram BCDE equal to the rectilineal figure A.



If then the sides of it, BE, ED, are equal to one another, it is a (30 *Def.*) square, and what was required is now done:

But if they are not equal, produce one of them BE to F, and make (3. I.) EF equal to ED, and bisect (10. I.) BF in G; and from the centre G, at the distance GB, or GF, describe the semicircle BHF, and produce DE to H, and join GH:

Therefore because the straight line BF is divided into two equal parts in the point G, and into two unequal at E, the rectangle BE, EF, together with the square of EG, is equal (5. II.) to the square of GF:

But GF is equal (15 *Def.*) to GH; therefore the rectangle BE, EF, together with the square of EG, is equal to the square of GH:

But the squares of HE, EG are equal (47. I.) to the square of GH; therefore the rectangle BE, EF, together with the square of EG, is equal to the squares of HE, EG:

Take away the square of EG, which is common to both, and the remaining rectangle BE, EF is equal (3 *Ax.*) to the square of EH:

But the rectangle contained by BE, EF is the parallelogram BD, because EF is equal to ED; therefore BD is equal to the square of EH:

But BD is equal (*Constr.*) to the rectilineal figure A; therefore the rectilineal figure A is equal to the square of EH. Wherefore a square has been made equal to the given rectilineal figure, A, viz. the square described upon EH. Which was to be done.

EXERCISES ON BOOK II.

1. (*a.*) The difference of the internal segments of a line, or the sum of the external, is double of the distance between the points of section and bisection.

   (*b.*) Also, that if half the sum of two lines be increased by half their difference, the sum will be equal to the greater of them; and if diminished by half their difference, the remainder will be equal to the less of them.

2. If two lines be each of them divided into any number of parts, equal or unequal to one another, the rectangle under the two lines is equal to all the rectangles taken together, under each separate part of the one with each separate part of the other.

   (*a.*) What will take place when all the divisions are equal?

3. Employ this to demonstrate 4 and 7, without first proving that the smaller squares are about the diagonal of the original square.

Show likewise that those propositions are respectively equivalent to these two:—

> (*a.*) The square on a line is greater than the squares described on its internal segments by twice the rectangle under those segments;
>
> (*b.*) The square on a line is less than the squares on its external segments by twice the rectangle under those segments.

4. The difference of the squares on two lines are equal to the rectangle under their sum and difference.

Show that Props. 4 and 5 are cases of this property.

5. The squares on the sum and difference of two lines are equal to twice the sum of the squares on the lines themselves.

6. The square on the sum of two lines exceeds the square on their difference by four times the rectangle under the lines themselves.

7. If a straight line be drawn from the vertex of a triangle to the middle of the base, the sum of the squares on the two sides is double the square on that line, together with double the square on half the base.

8. Three times the sum of the squares on the three sides of a triangle is equal to four times the sum of the squares of the lines drawn from the angles to bisect the opposite sides.

9. The difference between the squares on two sides of a triangle is equal to the difference between the squares on the segments of the base made by a perpendicular from the opposite angle, whether the segments be internal or external.

10. The sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on its four sides.

11. Lines being drawn from any point to the angles of a rectangle: the squares on those drawn to the extremities of one diagonal are together equal to the squares on those drawn to the other extremity.

12. Let the four sides and two diagonals of any quadrilateral be bisected: then

> (*a.*) The sum of the squares on the four sides of the quadrilateral will be equal to the sum of the squares on the two diagonals together with four times that of the line joining the middle points of the diagonals;
>
> (*b.*) The sum of the squares on either pair of opposite sides, together with the squares on the diagonals, is equal to the sum of the squares on the other pair of opposite sides, together with four times the square of the line which joins their middle points.

13. If two opposite sides of a quadrilateral be parallel, the sum of the squares on its diagonals is equal to the squares on the two sides which are not parallel together with twice the rectangle under the parallel sides.

14. If a line be drawn from the vertex of an isosceles triangle to meet the base or the base produced, the square upon one side of the triangle is equal to the square upon this line, increased or diminished

respectively by the rectangle under the segments of the base, as the segments are internal or external.

15. The greatest rectangle which can be made by the segments of a given line is the square described upon half that line.

16. In the figure to I. 47, show that,

(*a.*) $AD^2 - AE^2 = BA^2 - AC^2$;
(*b.*) $GH^2 + KE^2 + FD^2 = 6\ BC^2$:

the requisite lines being drawn.

BOOK III.

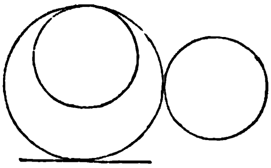
DEFINITIONS.

1. Equal circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

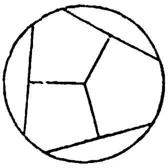
"This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal."

2. A straight line is said to touch a circle when it meets the circle, and being produced, does not cut it.

3. Circles are said to touch one another which meet but do not cut one another.

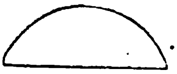


4. Straight lines are said to be equally distant from the centre of a circle when the perpendiculars drawn to them from the centre are equal.



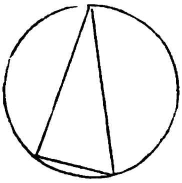
5. And the straight line on which the greater perpendicular falls is said to be farther from the centre.

6. A segment of a circle is the figure contained by a straight line and the circumference it cuts off.



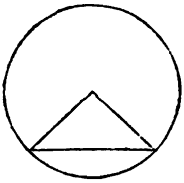
7. "The angle of a segment is that which is contained by the straight line and the circumference."

8. An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

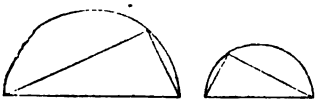


9. And an angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.

10. The sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



11. Similar segments of a circle are those in which the angles are equal, or which contain equal angles.



PROPOSITIONS.

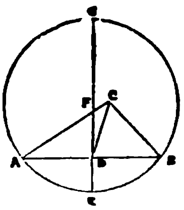
PROPOSITION I.

Prob. *To find the centre of a given circle.*

Let ABC be the given circle; it is required to find its centre.

Draw within it any straight line AB, and bisect (10. I.) it in D; from the point D draw (11. I.) DC at right angles to AB, and produce it to E, and bisect CE in F: the point F is the centre of the circle ABC.

For if it be not, let, if possible, G be the centre, and join GA, GD, GB: then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD, DG are equal to the two BD, DG, each to each; and the base GA is equal to the base GB,



because they are drawn from the centre G :* therefore the angle ADG is equal (8. I.) to the angle GDB: but when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle (10 *Def.* I.); therefore the angle GDB is a right angle:

But FDB is likewise a (*Constr.*) right angle; wherefore the angle FDB is equal (1 *Ax.*) to the angle GDB, the greater to the less, which is impossible: therefore G is not the centre of the circle ABC.

In the same manner it can be shown, that no other point but F is the centre; that is, F is the centre of the circle ABC. Which was to be found.

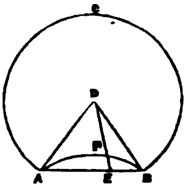
COR. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROPOSITION II.

THEOR. *If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.*

Let ABC be a circle, and A, B any two points in the circumference; the straight line drawn from A to B shall fall within the circle.

For if it do not, let it fall, if possible, without, as AEB: find (1. III.) D the centre of the circle ABC: and join AD, DB, and produce DF, any straight line meeting the circumference AB to E:



Then, because DA is equal (15 *Def.* I.) to DB, the angle DAB is equal (5. I.) to the angle DBA;

And because AE, a side of the triangle DAE, is produced to B, the angle DEB is greater (16. I.) than the angle DAE;

But DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE:

But to the greater angle the greater side is opposite (19. I.), DB is therefore greater than DE:

But DB is equal to DF; wherefore DF is greater than DE, the less than the greater, which is impossible: therefore the straight line drawn from A to B does not fall without the circle. In the same manner it may be demonstrated that it does not fall upon the circumference; it falls therefore within it. Wherefore, if any two points, etc. Q. E. D.

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* Whenever the expression "straight lines from the centre" or "drawn from the centre" occurs, it is to be understood that they are drawn to the circumference.

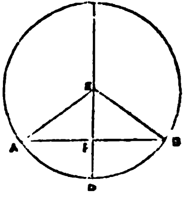
PROPOSITION III.

THEOR. *If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles: and if it cut it at right angles, it shall bisect it.*

Let ABC be a circle; and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre, in the point F: it cuts it also at right angles.

Take (1. III.) E the centre of the circle, and join EA, EB.

Then, because AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two sides in the one equal to two sides in the other, and the base EA is equal to the base EB; therefore the angle AFE is equal (8. I.) to the angle BFE:



But when a straight line standing upon another makes the adjacent angles equal to one another, each of them is a right (10 *Def.* I.) angle; therefore each of the angles AFE, BFE is a right angle: wherefore the straight line CD, drawn through the centre, bisecting another AB that does not pass through the centre, cuts the same at right angles.

But let CD cut AB at right angles: CD also bisects it, that is, AF is equal to FB.

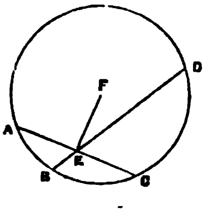
The same construction being made, because EA, EB from the centre are equal (15 *Def.* I.) to one another, the angle EAF is equal (5. I.) to the angle EBF; and the right angle AFE is equal to the right angle BFE: therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other; and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (26. I.); AF therefore is equal to FB. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION IV.

THEOR. *If in a circle two straight lines cut one another which do not both pass through the centre, they do not bisect each other.*

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another in the point E, and do not both pass through the centre: AC, BD do not bisect one another.

For, if it is possible, let AE be equal to EC, and BE to ED: if one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre:



But if neither of them pass through the centre, take (1. III.) F the centre of the circle, and join EF: and because FE, a straight line through the centre, bisects another AC which does not pass through the centre, it shall cut it at right (3. III.) angles; wherefore FEA is a right angle:

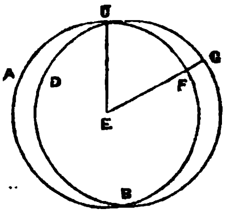
Again, because the straight line FE bisects the straight line BD which does not pass through the centre, it shall cut it at right (3. III.) angles; wherefore FEB is a right angle:

And FEA was shown to be a right angle: therefore FEA is equal (1 *Ax.*) to the angle FEB, the less to the greater, which is impossible: therefore AC, BD do not bisect one another. Wherefore, if in a circle, etc. Q. E. D.

PROPOSITION V.

THEOR. *If two circles cut one another, they shall not have the same centre.*

Let the two circles ABC, CDG cut one another in the points B, C: they have not the same centre.



For, if it be possible, let E be their centre; join EC, and draw any straight line EFG meeting them in F and G: and because E is the centre of the circle ABC, CE is equal (15 *Def.* I.) to EF:

Again, because E is the centre of the circle CDG, CE is equal (15 *Def.* I.) to EG:

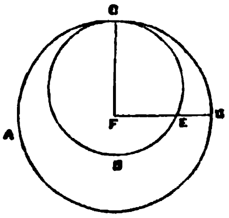
But CE was shown to be equal to EF; therefore EF is equal to EG, the less to the greater, which is impossible: therefore E is not the centre of the circles ABC, CDG. Wherefore, if two circles, etc. Q. E. D.

PROPOSITION VI.

THEOR. *If one circle touch another internally, they shall not have the same centre.*

Let the two circles ABC, CDE touch one another in the point C: they have not the same centre.

For if they have, let it be F; join FC, and draw any straight line FEB meeting them in E and B:



And because F is the centre of the circle ABC, CF is equal (15 *Def.* I.) to FB; also, because F is the centre of the circle CDE, CF is equal (15 *Def.* I.) to FE:

And CF was shown to be equal to FB; therefore FE is equal (1 *Ax.*) to FB, the less to the greater, which is impossible: wherefore F is not the centre of the circles ABC, CDE. Therefore, if two circles, etc. Q. E. D.

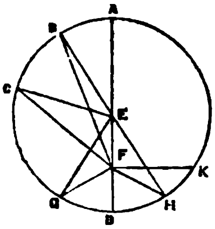
PROPOSITION VII.

THEOR. *If any point be taken in the diameter of a circle, which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote: and from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the shortest line.*

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre: let the centre be E; of all the straight lines FB, FC, FG, etc. that can be drawn from F to the circumference, FA is the greatest, and FD, the other part of the diameter AD, is the least: and of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE:

And because two sides of a triangle are greater (20. I.) than the third, BE, EF are greater than BF:



But AE is equal (15 *Def.* I.) to EB; therefore AE, EF, that is AF, is greater than BF:

Again, because BE is equal to CE, and FE common to the triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF; but the angle BEF is greater (9 *Ax.*) than the angle CEF; therefore the base BF is greater (24. I.) than the base FC:

For the same reason, CF is greater than GF:

Again, because GF, FE are greater (20. I.) than EG, and EG, is equal to ED; GF, FE are greater than ED: take away the common part FE, and the remainder GF is greater (5 *Ax.*) than the remainder FD:

Therefore FA is the greatest, and FD the least, of all the straight lines from F to the circumference; and BF is greater than CF, and CF than GF.

Also there can be drawn only two equal straight lines from the point F to the circumference, one upon each side of the shortest line FD: at the point E, in the straight line EF, make (23. I.) the angle FEH equal to the angle GEF, and join FH:

Then because GE is equal to EH, and EF common to the two triangles GEF, HEF, the two sides GE, EF are equal to the two HE, EF; and the angle GEF is equal (*Constr.*) to the angle HEF; therefore the base FG is equal (4. I.) to the base FH:

But, besides FH, no other straight line can be drawn from F to the circumference, equal to FG: for if there can, let it be FK: and because FK is equal to FG, and FG to FH, FK is equal (1. *Ax.*) to FH; that is, a line nearer to that which passes through the centre is equal to one which is more remote; which is impossible. Therefore, if any point be taken, etc. Q. E. D.

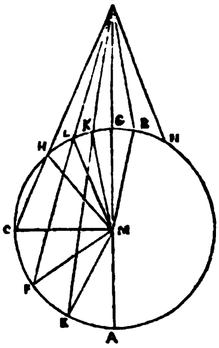
PROPOSITION VIII.

THEOR. *If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than the more remote; but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than the more remote: and only two equal straight lines can be drawn from the point to the circumference, one upon each side of the least.*

Let ABC be a circle, and D any point without it, from which let the straight lines DA, DE, DF, DC, be drawn to the circumference, whereof DA passes through the centre. Of those which fall upon the concave part of the circumference AEFC, the greatest is AD which passes through the centre; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC; but of those which fall upon the convex circumference HLKG, the least is DG, between the point D and the diameter AG; and the nearer to it is always less than the more remote, viz. DK than DL, and DL than DH.

Take (1. III.) M the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH:

And because AM is equal to ME, add MD to each, therefore AD is equal (*Ax.*) to EM, MD:



But EM, MD are greater (20. I.) than ED; therefore also AD is greater than ED.

Again, because ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD: but the angle EMD is greater (9 *Ax.*) than the angle FMD; therefore the base ED is greater (24. I.) than the base FD.

In like manner it may be shown that FD is greater than CD: therefore DA is the greatest: and DE greater than DF, and DF than DC.

And because MK, KD are greater (20. I.) than MD, and MK is equal (15 *Def.* I.) to MG, the remainder DK is greater (5 *Ax.*) than the remainder GD; that is, GD is less than KD:

And because MK, DK are drawn to the point K within the triangle MLD, from M, D, the extremities of its side MD, MK, KD are less (21. I.) than ML, LD, whereof MK is equal to ML; therefore the remainder DK is less than the remainder DL:

In like manner it may be shown, that DL is less than DH: therefore DG is the least, and DK less than DL, and DL than DH.

Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least.

At the point M, in the straight line MD, make (23. I.) the angle DMB equal to the angle DMK, and join DB:

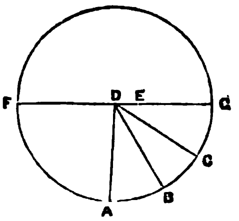
And because MK is equal to MB, and MD common to the triangles KMD, BMD, the two sides KM, MD are equal to the two BM, MD; and the angle KMD is equal (*Constr.*) to the angle BMD; therefore the base DK is equal (4. I.) to the base DB:

But, besides DB, there can be no straight line drawn from D to the circumference equal to DK: for if there can, let it be DN: and because DK is equal to DN, and also to DB, therefore DB is equal to DN; that is, the nearer to the least equal to the more remote, which is impossible. If therefore any point, etc. Q. E. D.

PROPOSITION IX.

THEOR. *If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.*

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal straight lines, viz. DA, DB, DC: the point D is the centre of the circle.



For, if not, let E be the centre; join DE, and produce it to the circumference in F, G;

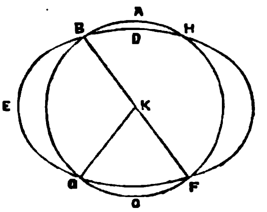
Then FG is a diameter of the circle ABC; and because in FG, the diameter of the circle ABC, there is taken the point D, which is not the centre, DG shall be the greatest line from it to the circumference, and DC greater (7. III.) than DB, and DB than DA:

But they are likewise equal (*Hyp.*), which is impossible; therefore E is not the centre of the circle ABC: in like manner it may be demonstrated, that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, etc. Q. E. D.

PROPOSITION X.

THEOR. *One circumference of a circle cannot cut another in more than two points.*

If it be possible, let the circumference FAB cut the circumference DEF in more than two points, viz., in B, G, F: take the centre K of the circle ABC, and join KB, KG, KF:



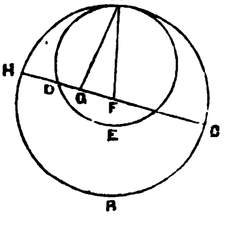
And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, the point K is (9. III.) the centre of the circle DEF:

But K is also the centre (*Constr.*) of the circle ABC; therefore the same point is the centre of two circles that cut one another, which is impossible (5. III.). Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

PROPOSITION XI.

THEOR. *If one circle touch another internally, the straight line which joins their centres, being produced, shall pass through the point of contact.*

Let the two circles ABC, ADE touch each other in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE: the straight line which joins the centres F, G, being produced, passes through the point A.



For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG.

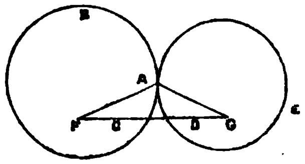
And because AG, GF are greater (20. I.) than FA, that is, than FH, for FA is equal (15 *Def.* I.) to FH; take away the common part FG; therefore the remainder AG is greater than the remainder GH:

But AG is equal to GD; therefore GD is greater than GH, the less than the greater, which is impossible. Therefore the straight line which joins the points F, G cannot fall otherwise than upon the point

A, that is, it must pass through it. Therefore, if two circles, etc. Q. E. D.

PROPOSITION XII.

THEOR. *If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.*



Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: the straight line which joins the points F, G shall pass through the point of contact A.

For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG.

And because F is the centre of the circle ABC, AF is equal to FC:

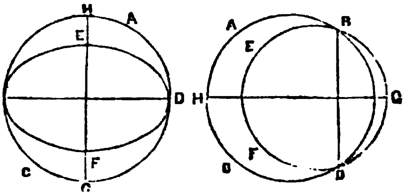
Also, because G is the centre of the circle ADE, AG is equal to GD:

Therefore FA, AG are equal (2 *Ax.*) to FC, DG; wherefore the whole FG is greater than FA, AG: but it is also less (20. I.), which is impossible: therefore the straight line which joins the points F, G shall not pass otherwise than through the point of contact A, that is, it must pass through it. Therefore, if two circles, etc. Q. E. D.

PROPOSITION XIII.

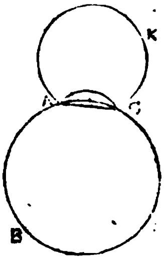
THEOR. *One circle cannot touch another in more points than one, whether it touches it on the inside or outside.*

For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first on the inside, in the points B, D; join BD, and draw (11. I.) GH bisecting BD at right angles·



Therefore, because the points B, D are in the circumference of each of the circles, the straight line BD falls within each (2. III.) of them: and their centres are (1. III.) in the straight line GH which bisects BD at right angles: therefore GH passes through the point of contact (11. III.);

But it does not pass through it, because the points B, D are without the straight line GH, which is absurd: therefore one circle cannot touch another on the inside in more points than one.



Nor can two circles touch one another on the outside in more than one point; for, if it be possible, let the circle ACK touch the circle ABC in the points A, C, and join AC:

Therefore, because the two points A, C are in the circumference of the circle ACK, the straight line AC which joins them shall fall within (2. III.) the circle ACK:

And the circle ACK is without the circle ABC; and therefore the straight line AC is without (*Hyp.*) this last circle:

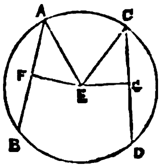
But because the points A, C are in the circumference of the circle ABC, the straight line AC must be within (2. III.) the same circle, which is absurd; therefore one circle cannot touch another on the outside in more than one point: and it has been shown that they cannot touch on the inside in more points than one. Therefore, one circle, etc. Q. E. D.

PROPOSITION XIV.

THEOR. *Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.*

Let the straight lines AB, CD, in the circle ABDC, be equal to one another; they are equally distant from the centre.

Take (1. III.) E, the centre of the circle ABDC, and from it (12. I.) draw EF, EG, perpendiculars to AB, CD:



Then, because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects (3. III.) it: wherefore AF is equal to FB, and AB double of AF.

For the same reason, CD is double of CG;

And AB is equal (*Hyp.*) to CD; therefore AF is equal (7 *Ax.*) to CG:

And because AE is equal (15 *Def.* I.) to EC, the square of AE is equal to the square of EC;

But the squares of AF, FE are equal (47. I.) to the square of AE, because the angle AFE is a right angle; and for the like reason, the squares of EG, GC are equal to the square of EC:

Therefore the squares of AF, FE are equal (1 *Ax.*) to the squares of CG, GE, of which the square of AF is equal to the square of CG, because AF is equal to CG; therefore the remaining square of FE is equal (3 *Ax.*) to the remaining square of EG, and the straight line EF is therefore equal to EG;

But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (4 *Def.* III.); therefore AB, CD must be equally distant from the centre.

Next, if the straight lines AB, CD be equally distant from the centre, that is, if FE be equal to EG; AB is equal to CD:

For, the same construction being made, it may, as before, be demonstrated, that AB is double of AF, and CD double of CG, and that the squares of EF, FA are equal to the squares of EG, GC; of which the square of FE is equal to the square of EG, because FE is equal to EG; therefore the remaining square of AF is equal (3 *Ax.*) to the remaining square of CG; and the straight line AF is therefore equal to CG;

And AB is double of AF, and CD double of CG: wherefore AB is equal (6 *Ax.*) to CD. Therefore equal straight lines, etc. Q. E. D.

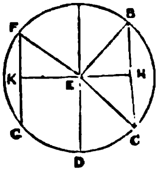
PROPOSITION XV.

THEOR. *The diameter is the greatest straight-line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote: and the greater is nearer to the centre than the less.*

Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG: AD is greater than any straight line BC, which is not a diameter, and BC greater than FG.

From the centre draw (12. I.) EH, EK, perpendiculars to BC, FG, and join EB, EC, EF;

And because AE is equal (15 *Def.* I.) to EB, and ED to EC, AD is equal to EB, EC:



But EB, EC are greater (20. I.) than BC; wherefore also AD is greater than BC.

And, because BC is nearer to the centre than FG, EH is less (5 *Def.* III.) than EK:

But, as was demonstrated in the preceding, BC is double of BH, and FG double of FK, and the squares of EH, HB are equal to the squares EK, KF, of which the square of EH is less than the square of EK, because EH is less than EK; therefore the square of BH is greater than the square of FK, and the straight line BH greater than FK, and therefore BC is greater than FG.

Next, let BC be greater than FG: BC is nearer to the centre than FG, that is, the same construction being made, EH is less than EK.

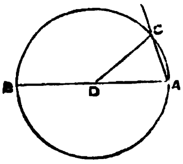
Because BC is greater than FG, BH likewise is greater than KF; and the squares of BH, HE are equal to the squares of FK, KE, of which the square of BH is greater than the square of FK, because BH is greater than FK; therefore the square of EH is less than the square of EK, and the straight line EH less than EK. Wherefore the diameter, etc. Q. E. D.

PROPOSITION XVI.

THEOR. *The straight line drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle; and no straight line can be drawn between that straight line and the circumference from the extremity, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.*

Let ABC be a circle, the centre of which is D, and the diameter AB: the straight line drawn at right angles to AB from its extremity A, shall fall without the circle.

For, if it does not, let it fall, if possible, within the circle, as AC, and draw DC to the point C, where it meets the circumference:

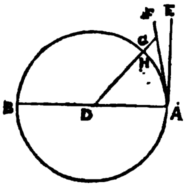


And because DA is equal (15 *Def.* I.) to DC, the angle DAC is equal (5. I.) to the angle ACD: but DAC is a right (*Hyp.*) angle; therefore ACD is a right angle, and the angles DAC, ACD are therefore equal to two right angles, which is impossible (17. I.); therefore the straight line drawn from A at right angles to BA does not fall within the circle.

In the same manner it may be demonstrated, that it does not fall upon the circumference; therefore it must fall without the circle, as AE.

And between the straight line AE and the circumference no straight line can be drawn from the point A which does not cut the circle: for, if possible, let FA be between them, and from the point D draw (12. I.) DG perpendicular to FA, and let it meet the circumference in H:

And because AGD (*Constr.*) is a right angle, and DAG less (17. I.) than a right angle, DA is greater (19. I.) than DG:



But DA is equal (15 *Def.* I.) to DH; therefore DH is greater than DG, the less than the greater, which is impossible: therefore no straight line can be drawn from the point A between AE and the circumference which does not cut the circle; or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point A, or however small an angle it makes with AE, the circumference passes between that straight line and the perpendicular AE. "And this is all that is to be understood, when, in the Greek text, and translations from it, the angle of the semicircle is said to be greater than any acute rectilineal angle, and the remaining angle less than any rectilineal angle." Q. E. D.

COR. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because, if it did meet the circle in two, it would fall within it (2. III.). "Also, it is evident that there can be but one straight line which touches the circle in the same point."

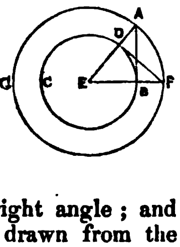
PROPOSITION XVII.

PROB. *To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.*

First, let A be a given point without the given circle BCD; it is required to draw a straight line from A which shall touch the circle.

Find (1. III.) the centre E of the circle, and join AE; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw (11. I.) DF at right angles to EA, and join EBF, AB: AB touches the circle BCD.

Because E is the centre of the circles BCD, AFG, EA is equal (15 *Def.* I.) to EF, and ED to EB:



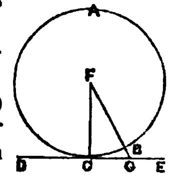
Therefore the two sides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the base DF is equal to the base AB; and the triangle FED to the triangle AEB, and the other angles to the other angles (4. I.); therefore the angle EBA is equal to the angle EDF:

But EDF is a right angle, wherefore EBA is a right angle; and EB is drawn from the centre: but a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle (*Cor.* 16. III.): therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.

But if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE: DF touches the circle (*Cor.* 16. III.).

PROPOSITION XVIII.

THEOR. *If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.*



Let the straight line DE touch the circle ABC in the point C; take the centre F, and draw the straight line FC: FC is perpendicular to DE.

For if it be not, from the point F draw FBG perpendicular to DE;

And because FGC is a right angle, GCF is (17. I.) an acute angle; and to the greater angle the greater (19. I.) side is opposite: therefore FC is greater than FG:

But FC is equal (15 *Def.* I.) to FB; therefore FB is greater than FG, the less than the greater, which is impossible: wherefore FG is not perpendicular to DE.

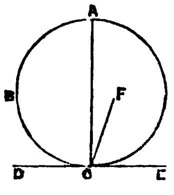
In the same manner it may be shown, that no other is perpendicular to it besides FC, that is, FC is perpendicular to DE. Therefore, if a straight line, etc. Q. E. D.

PROPOSITION XIX.

THEOR. *If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.*

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.

For, if not, let F be the centre, if possible, and join CF.



Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular (18. III.) to DE; therefore FCE is a right angle.

But ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible; wherefore F is not the centre of the circle ABC.

In the same manner it may be shown, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a straight line, etc. Q. E. D.

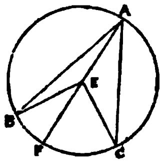
PROPOSITION XX.

THEOR. *The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.*

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same circumference BC for their base; the angle BEC is double of the angle BAC.

First, let E, the centre of the circle, be within the angle BAC, and join AE, and produce it to F:

Because EA is equal to EB, the angle EAB is equal (5. I.) to the angle EBA; therefore the angles EAB, EBA are double of the angle EAB:



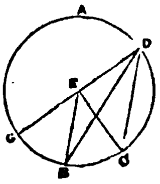
But the angle BEF is equal (32. I.) to the angles EAB, EBA; therefore also the angle BEF is double of the angle EAB:

For the same reason, the angle FEC is double of the angle EAC;

Therefore the whole angle BEC is double of the whole angle BAC.

Again, let E, the centre of the circle, be without the angle BDC, and join DE, and produce it to G.

It may be demonstrated, as in the first case, that the angle GEC is double of the angle GDC, and that GEB, a part of the first, is double of GDB, a part of the other; therefore the remaining angle BEC is double of the remaining angle BDC. Therefore the angle at the centre, etc. Q. E. D.



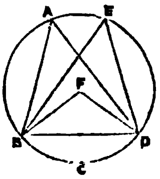
PROPOSITION XXI.

THEOR. *The angles in the same segment of a circle are equal to one another.*

Let ABCD be a circle, and BAD, BED angles in the same segment BAED: the angles BAD, BED are equal to one another.

Take (1. III.) F, the centre of the circle ABCD: and, first, let the segment BAED be greater than a semicircle, and join BF, FD:

And because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same part of the circumference, viz. BCD for their base; therefore the angle BFD is double (20. III.) of the angle BAD:

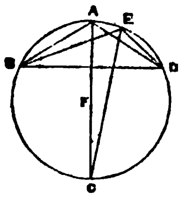


For the same reason, the angle BFD is double of the angle BED;

Therefore the angle BAD is equal to the angle BED.

But, if the segment BAED be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another; draw AF to the centre, and produce it to C, and join CE.

Therefore the segment BADC is greater than a semicircle; and the angles in it, BAC, BEC, are equal, by the first case:



For the same reason, because CBED is greater than a semicircle, the angles CAD, CED are equal;

Therefore the whole angle BAD is equal to the whole angle BED. Wherefore, the angles in the same segment, etc. Q. E. D.

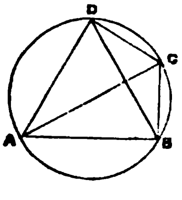
PROPOSITION XXII.

THEOR. *The opposite angles of any quadrilateral figure inscribed in a circle, are together equal to two right angles.*

Let ABCD be a quadrilateral figure in the circle ABCD: any two of its opposite angles are together equal to two right angles.

Join AC, BD:

And because the three angles of every triangle are equal (32. I.) to two right angles, the three angles of the triangle CAB, viz., the angles CAB, ABC, BCA are equal to two right angles:



But the angle CAB is equal (21. III.) to the angle CDB, because they are in the same segment BADC; and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB; therefore the whole angle ADC is equal to the angles CAB, ACB:

To each of these equals add the angle ABC; therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC:

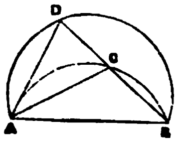
But ABC, CAB, BCA are equal to two right angles; therefore also the angles ABC, ADC are equal to two right angles.

In the same manner the angles BAD, DCB may be shown to be equal to two right angles. Therefore, the opposite angles, etc. Q. E. D.

PROPOSITION XXIII.

THEOR. *Upon the same straight line, and upon the same side of it, there cannot be two similar segments of circles, not coinciding with one another.*

If it be possible, let the two similar segments of circles, viz. ACB, ADB be upon the same side of the same straight line AB, not coinciding with one another. Then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (10. III.); one of the segments must therefore fall within the other: let ACB fall within ADB, and draw the straight line BCD, and join CA, DA:



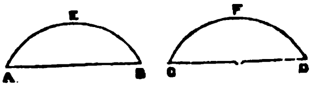
And because the segment ACB is similar to the segment ADB, and that similar segments of circles contain (11 *Def.* III.) equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impossible (16. I.). Therefore, there cannot be two similar segments of circles upon the same side of the same line which do not coincide. Q. E. D.

PROPOSITION XXIV.

THEOR. *Similar segments of circles upon equal straight lines are equal to one another.*

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD: the segment AEB is equal to the segment CFD.

For if the segment AEB be applied to the segment CFD, so as the point A be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD:



Therefore the straight line AB coinciding with CD, the segment AEB must (23. III.) coincide with the segment CFD, and therefore is equal to it. Wherefore, similar segments, etc. Q. E. D.

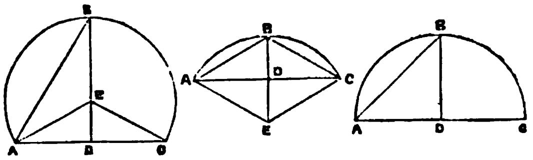
PROPOSITION XXV.

PROB. *A segment of a circle being given, to describe the circle of which it is the segment.*

Let ABC be the given segment of a circle: it is required to describe the circle of which it is the segment.

Bisect (10. I.) AC in D, and from the point D draw (11. I.) DB at right angles to AC, and join AB.

First, let the angles ABD, BAD be equal to one another; then the straight line BD is equal (6. I.) to DA, and therefore to DC: and because the three straight lines DA, DB, DC are all equal, D is the centre of the circle (9. III.). From the centre D, at the distance of any of the three DA, DB, DC, describe a circle; this shall pass through the other points, and the circle, of which ABC is a segment, is described: and because the centre D is in AC, the segment ABC is a semicircle.



But if the angles ABD, BAD are not equal to one another, at the point A, in the straight line AB, make (23. I.) the angle BAE equal to the angle ABD, and produce BD, if necessary, to E, and join EC:

And because the angle ABE is equal to the angle BAE, the straight line BE is equal (6. I.) to EA:

And because AD is equal to DC, and DE common to the triangles ADE, CDE, the two sides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE for each of them is a right (*Constr.*) angle; therefore the base AE is equal (4. I.) to the base EC:

But AE was shown to be equal to EB, wherefore also BE is equal to EC: and the three straight lines AE, EB, EC are therefore equal to one another; wherefore (9. III.) E is the centre of the circle. From the centre E, at the distance of any of the three AE, EB, EC, describe a circle; this shall pass through the other points; and the circle, of which ABC is a segment, is described.

And it is evident, that if the angle ABD be greater than the angle

BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle: but if the angle ABD be less than BAD, the center E falls within the segment ABC, which is therefore greater than a semicircle. Wherefore, a segment of a circle being given, the circle is described of which it is a segment. Which was to be done.

PROPOSITION XXVI.

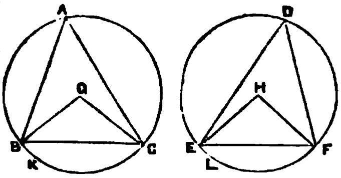
THEOR. *In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.*

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences: the circumference BKC is equal to the circumference ELF.

Join BC, EF:

And because the circles ABC, DEF are equal (*Hyp.*), the straight lines drawn from their centres are equal (1 *Def.* III.): therefore the two sides BG, GC are equal to the two EH, HF: and the angle at G is equal to the angle at H; therefore the base BC is equal (4. I.) to the base EF.

And because the angle at A is equal to the angle at D, the segment BAC is similar (11 *Def.* III.) to the segment EDF; and they are upon equal straight lines BC, EF:



But similar segments of circles upon equal straight lines are equal (24. III.) to one another; therefore the segment BAC is equal to the segment EDF:

But the whole circle ABC is equal (*Hyp.*) to the whole DEF; therefore the remaining segment BKC is equal (3 *Ax.*) to the remaining segment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, etc. Q. E. D.

PROPOSITION XXVII.

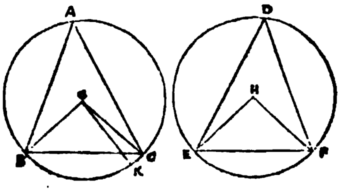
THEOR. *In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.*

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF, stand upon the equal circumferences BC, EF: the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifest (20. III.) that the angle BAC is also equal to EDF.

But, if not, one of them is the greater. Let BGC be the greater, and at the point G, in the straight line BG, make (23. I.) the angle BGK equal to the angle EHF.

Now equal angles stand upon equal circumferences (26. III.), when they are at the centre; therefore the circumference BK is equal to the circumference EF:



But EF is equal to BC; therefore also BK is equal to BC, the less to the greater, which is impossible;

Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it:

And the angle at A is half of the angle BGC, and the angle at D half of the angle EHF; therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, etc. Q. E. D.

PROPOSITION XXVIII.

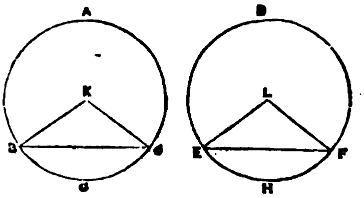
THEOR. *In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.*

Let ABC, DEF be equal circles, and BC, EF equal straight lines in them, which cut off the two greater circumferences BAC, EDF, and the two less BGC, EHF: the greater BAC is equal to the greater EDF, and the less BGC to the less EHF.

Take (1. III.) K, L, the centres of the circles, and join BK, KC, EL, LF:

And because the circles are equal (*Hyp.*), the straight lines from their centres are equal; therefore BK, KC are equal to EL, LF; and the base BC is equal to the base EF; therefore the angle BKC is equal (8. I.) to the angle ELF:

But equal angles stand upon equal (26. III.) circumferences, when they are at the centres; therefore the circumference BGC is equal to the circumference EHF:



But the whole circle ABC is equal (*Hyp.*) to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal (3 *Ax.*) to the remaining part EDF. Therefore, in equal circles, etc. Q. E. D.

PROPOSITION XXIX.

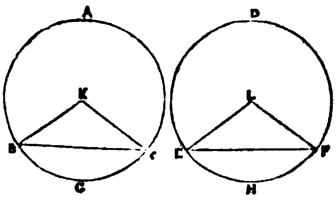
THEOR. *In equal circles, equal circumferences are subtended by equal straight lines.*

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF also be equal; and join BC, EF: the straight line BC is equal to the straight line EF.

Take (1. III.) K, L, the centres of the circles, and join BK, KC; EL, LF:

And because the circumference BGC is equal to the circumference EHF, the angle BKC is equal (27. III.) to the angle ELF:

And because the circles ABC, DEF are equal, the straight lines from their centres are equal; therefore BK, KC are equal to EL, LF, and they contain equal angles; therefore the base BC is equal (4. I.) to the base EF. Therefore, in equal circles, etc. Q. E. D.



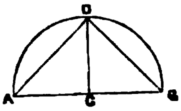
PROPOSITION XXX.

Prob. *To bisect a given circumference*, that is, to divide it into two equal parts.

Let ADB be the given circumference; it is required to bisect it.

Join AB, and bisect (10. I.) it in C; from the point C draw CD at right angles (11. I.) to AB, and join AD, DB: the circumference ADB is bisected in the point D.

Because AC is equal to CB, and CD common to the triangles ACD, BCD, the two sides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, because each of them is a right (*Constr.*) angle; therefore the base AD is equal (4. I.) to the base BD.



But equal straight lines cut off equal (28. III.) circumferences, the greater equal to the greater, and the less to the less; and AD, DB are each of them less than a semicircle, because DC passes through the centre (*Cor.* 1. III.); wherefore the circumference AD is equal to the circumference DB. Therefore the given circumference is bisected in D. Which was to be done.

PROPOSITION XXXI.

Theor. *In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.*

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC: the angle in the semicircle BAC is a right angle; and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.

Join AE, and produce BA to F:

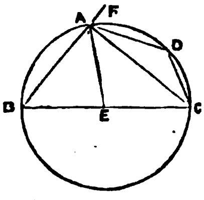
And because BE is equal (15 *Def.* I.) to EA, the angle EAB is equal (5. I.) to EBA;

Also, because AE is equal to EC, the angle EAC is equal to ECA;

Wherefore the whole angle BAC is equal to the two angles ABC, ACB:

But FAC, the exterior angle of the triangle ABC, is equal (32. I.) to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right (10 *Def.* I.) angle: wherefore the angle BAC in a semicircle is a right angle.

And because the two angles ABC, BAC of the triangle ABC are together less (17. I.) than two right angles, and that BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC greater than a semicircle, is less than a right angle.



And because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are equal (22. III.) to two right angles; therefore the angles ABC, ADC are equal to two right angles: and ABC is less than a right angle; wherefore the other ADC is greater than a right angle.

Besides, it is manifest that the circumference of the greater segment ABC falls without the right angle CAB; but the circumference of the less segment ADC falls within the right angle CAF. 'And this is all that is meant, when in the Greek text, and the translations from it, the angle of the greater segment is said to be greater, and the angle of the less segment is said to be less, than a right angle.'

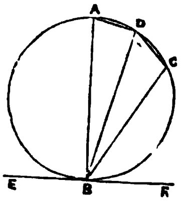
COR. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two; and when the adjacent angles are equal, they are right angles.

PROPOSITION XXXII.

THEOR. *If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.*

Let the straight line EF touch the circle ABCD in B, and from the point B let the straight line BD be drawn, cutting the circle: the angles which BD makes with the touching line EF shall be equal to the angles in the alternate segments of the circle; that is, the angle FBD is equal to the angle which is in the segment DAB, and the angle DBE to the angle in the segment BCD.

From the point B draw (11. I.) BA at right angles to EF, and take any point C in the circumference BD, and join AD, DC, CB:



And because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the centre of the circle is (19. III.) in BA; therefore the angle ADB in a semicircle is a right (31. III.) angle; and consequently the other two angles BAD, ABD are equal to a right angle:

But ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD, ABD:

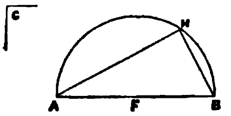
Take from these equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate segment of the circle.

And because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are equal (22. III.) to two right angles; therefore the angles DBF, DBE, being likewise equal (13. I.) to two right angles, are equal to the angles BAD, BCD: and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle. Wherefore, if a straight line, etc. Q. E. D.

PROPOSITION XXXIII.

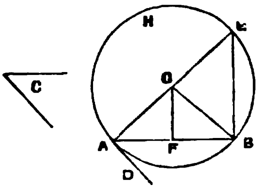
PROB. *Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.*

Let AB be the given straight line, and the angle at C the given rectilineal angle; it is required to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.



First, let the angle at C be a right angle; and bisect (10. I.) AB in F, and from the centre F, at the distance FB, describe the semicircle AHB; therefore the angle AHB in a semicircle is (31. III.) equal to the right angle at C.

But if the angle C be not a right angle, at the point A, in the straight line AB, make (23. I.) the angle BAD equal to the angle C, and from the point A draw (11. I.) AE at right angles to AD: bisect (10. I.) AB in F, and from F draw (11. I.) FG at right angles to AB, and join GB.

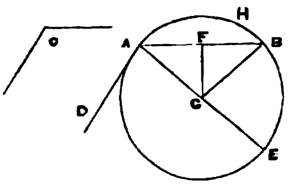


And because AF is equal to FB, and FG common to the triangles AFG, BFG, the two sides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal (4. I.) to the base GB; and the circle described from the centre G, at the distance GA, shall pass through the point B:

Let this be the circle AHB: and because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD (*Cor.* 16. III.) touches the circle:

And because AB, drawn from the point of contact A, cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (32. III.).

But the angle DAB is equal to the angle C; therefore also the angle C is equal to the angle in the segment AHB. Wherefore, upon the given straight line AB, the segment AHB of a circle is described, which contains an angle equal to the given angle at C. Which was to be done.



PROPOSITION XXXIV.

PROB. *To cut off a segment from a given circle, which shall contain an angle equal to a given rectilineal angle.*

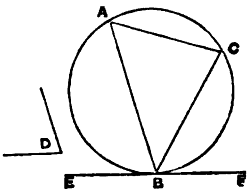
Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a segment from the circle ABC that shall contain an angle equal to the given angle D.

Draw (17. III.) the straight line EF touching the circle ABC in the point B; and at the point B, in the straight line BF, make (23. I.) the angle FBC equal to the angle D:

Therefore, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal

(32. III.) to the angle in the alternate segment BAC of the circle:

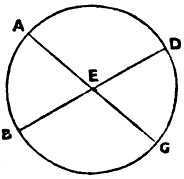
But the angle FBC is equal to the angle D; therefore the angle in the segment BAC is equal to the angle D. Wherefore the segment BAC is cut off from the given circle ABC, containing an angle equal to the given angle D. Which was to be done.



PROPOSITION XXXV.

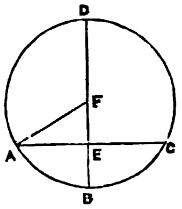
Theor. *If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.*

Let the two straight lines AC, BD, within the circle ABCD, cut one another in the point E; the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.



If AC, BD pass each of them through the centre, so that E is the centre, it is evident that AE, EC, BE, ED being all (15 *Def.* 1) equal, the rectangle AE, EC is likewise equal to the rectangle BE, ED.

But let one of them, BD, pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, in the point E: then, if BD be bisected in F, F is the centre of the circle ABCD; join AF



And because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E, AE, EC are equal (3. III.) to one another:

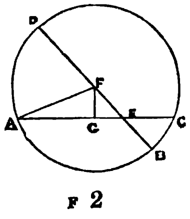
And because the straight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED, together with the square of EF, is equal (5. II.) to the square of FB; that is, to the square of FA:

But the squares of AE, EF are equal (47. I.) to the square of FA; therefore the rectangle BE, ED, together with the square of EF, is equal to the squares of AE, EF:

Take away the common square of EF, and the remaining rectangle BE, ED is equal (3 *Ax.*) to the remaining square of AE; that is, to the rectangle AE, EC.

Next, let BD, which passes through the centre, cut the other AC, which does not pass through the centre in E, but not at right angles: then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw (12. I.) FG perpendicular to AC;

Therefore AG is equal (3. III.) to GC; wherefore the rectangle AE, EC, together with the square of EG, is equal (5. II.) to the square of AG:



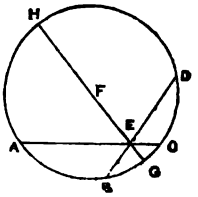
To each of these equals add the square of GF; therefore the rectangle AE, EC, together with the squares of EG, GF, is equal to the squares of AG, GF:

But the squares of EG, GF are equal (47. I.) to the square of EF; and the squares of AG, GF are equal to the square of AF; therefore the rectangle AE, EC, together with the square of EF, is equal to the square of AF; that is, to the square of FB:

But the square of FB is equal (5. II.) to the rectangle BE, ED, together with the square of EF; therefore the rectangle AE, EC, together with the square of EF, is equal to the rectangle BE, ED, together with the square of EF:

Take away the common square of EF, and the remaining rectangle AE, EC is therefore equal to the remaining rectangle BE, ED.

Lastly, let neither of the straight lines AC, BD pass through the centre: take (1. III.) the centre F, and through E, the intersection of the straight lines AC, DB, draw the diameter GEFH:



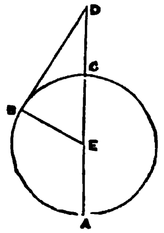
And because the rectangle AE, EC is equal, as has been shown, to the rectangle GE, EH; and, for the same reason, the rectangle BE, ED is equal to the same rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two straight lines, etc. Q. E. D.

PROPOSITION XXXVI.

THEOR. *If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.*

Let D be any point without the circle ABC, and DCA, DB two straight lines drawn from it, of which DCA cuts the circle, and DB touches the same: the rectangle AD, DC is equal to the square of DB.

Either DCA passes through the centre, or it does not: first, let it pass through the centre E, and join EB; therefore the angle EBD is a right (18. III.) angle:



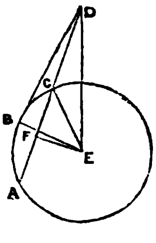
And because the straight line AC is bisected in E, and produced to the point D, the rectangle AD, DC, together with the square of EC, is equal (6. II.) to the square of ED: and CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED:

But the square of ED is equal (47. I.) to the squares of EB, BD, because EBD is a right angle; therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD:

Take away the common square of EB; therefore the remaining rectangle AD, DC is equal to the square of the tangent DB.

But if DCA does not pass through the centre of the circle ABC, take (1. III.) the centre E, and draw EF perpendicular (12. I.) to AC, and join EB, EC, ED.

And because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at



right angles, it shall likewise bisect (3. III.) it; therefore AF is equal to FC:

And because the straight line AC is bisected in F, and produced to D, the rectangle AD, DC, together with the square of FC, is equal (6. II.) to the square of FD:

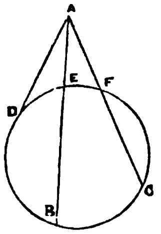
To each of these equals add the square of FE; therefore the rectangle AD, DC, together with the squares of CF, FE, is equal to the squares of DF, FE:

But the square of ED is equal (47. I.) to the squares of DF, FE, because EFD is a right angle: and the square of EC is equal to the squares of CF, FE; therefore the rectangle AD, DC, together with the square of EC, is equal to the square of ED:

And CE is equal to EB; therefore the rectangle AD, DC, together with the square of EB, is equal to the square of ED:

But the squares of EB, BD are equal to the square (47. 1.) of ED, because EBD is a right angle; therefore the rectangle AD, DC, together with the square of EB, is equal to the squares of EB, BD:

Take away the common square of EB; therefore the remaining rectangle AD, DC is equal to the square of DB. Wherefore, if from any point, etc. Q. E. D.



Cor. If from any point without a circle there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle are equal to one another, viz., the rectangle BA, AE, to the rectangle CA, AF; for each of them is equal to the square of the straight line AD, which touches the circle.

PROPOSITION XXXVII.

Theor. *If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the square of the line which meets it, the line which meets shall touch the circle.*

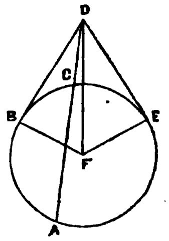
Let any point D be taken without the circle ABC, and from it let two straight lines DCA and DB be drawn of which DCA cuts the circle, and DB meets it: if the rectangle AD, DC be equal to the square of DB, DB touches the circle.

Draw (17. III.) the straight line DE, touching the circle ABC; find (1. III.) its centre F, and join FE, FB, FD; then FED is a right (18. III.) angle:

And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal (36. III.) to the square of DE:

But the rectangle AD, DC is, by hypothesis, equal to the square of DB; therefore the square of DE is equal to the square of DB:

And the straight line DE equal to the straight line DB: and FE is



equal to FB; wherefore DE, EF are equal to DB, BF; and the base FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal (8. I.) to the angle DBF:

But DEF is a right angle; therefore also DBF is a right angle: and FB, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches (16. III.) the circle: therefore DB touches the circle ABC. Wherefore, if from a point, etc. Q. E. D.

EXERCISES ON BOOK III.

1. Let two parallel lines meet a circle: then,

(*a.*) If these both touch the circle, the points of contact are at the opposite extremities of a diameter;

(*b.*) If one cut and the other touch, the first cuts off equal arcs of the circle estimated from the point of contact of the second;

(*c.*) If both cut, they intercept equal arcs of the circle between them;

(*d, e, f.*) State and prove the converses of these propositions.

2. If two lines be oblique to each other, and parallels to them be drawn to cut a circle: then,

(*a.*) The sum of the two opposite arcs, intercepted between these two lines, will be always the same, wherever the point of intersection may be, provided it be *within* the circle;

(*b.*) The difference will be the same wherever the point of intersection be taken *without* the circle;

(*c, d.*) State and prove the converses.

3. Two triangles ABC, A'B'C' are inscribed in the same circle, and the sides AB, AC are respectively parallel to A'B', A'C'; show that B'C is parallel to BC', and BB' equal to CC'.

4. Two chords of a circle cut one another and make equal angles with the diameter through the point of intersection: these chords will be equal.

5. If two chords AB, CD cut one another at right angles at E: then,

(*a.*) The sum of the arcs AC, BD are together equal to AD, BC; and each pair together equal to a semicircle;

(*b.*) The sum of the squares on AE, BE, CE, DE, is equal to the square on the diameter.

6. (*a.*) Find the least and greatest chord that can be drawn through a given point within a circle:—

(*b.*) Through a given point within or without a circle draw a chord to cut off a quadrant:—

(*c.*) Also through that point draw a chord of given length; and state the limitations of position to which that point must be subject.

7. In a given line to find a point such that tangents drawn from it to a given circle shall contain a given angle. Is this capable of one or more solutions?

8. If equal chords be drawn in a circle, they will all be tangents to another circle concentric with the former; and these chords will all be bisected at their points of contact.

Conversely, if chords be drawn in the outer of two concentric circles to touch the inner, these will all be equal, and be bisected at the points of contact.

9. Given the base and vertical angle of a triangle to construct it in the several cases where the third datum is:—

(*a.*) The perpendicular;
(*b.*) The sum of the sides;
(*c.*) The difference of the sides;
(*d.*) One of the angles at the base;
(*e.*) The difference of the angles at the base;
(*f.*) A line drawn from the vertex to a given point in the base.

10. Given the angles at the base of a triangle and the radius of the circumscribing circle: to construct the triangle; and to point out which of the cases of the preceding exercise is virtually identical with this.

11. Except one pair of opposite angles together be equal to the other pair together in a quadrilateral, a circle cannot be described about the quadrilateral; but if that condition be fulfilled, it always can.

12. No parallelogram but a rectangle can be inscribed in a circle.

13. If two circles touch each other, any straight line drawn through the point of contact will divide them into segments, of which the greater is similar to the greater and the less to the less; and the diameters through the points of intersection of the line with the circles will be parallel.

14. Through two given points to describe a circle which shall bisect the circumference of a given circle.

15. If all the interior or all the exterior angles of a quadrilateral be bisected, and the four bisecting lines in each case form a new quadrilateral: then each of these quadrilaterals will be capable of circumscription by a circle.

16. If about each of the triangles into which a quadrilateral is divided by its diagonals, circles be described; their centres will be the angular points of a parallelogram.

What takes place when the given quadrilateral is inscriptible in a circle; and when it is a parallelogram?

17. Let the opposite sides of a quadrilateral be produced to meet, and circles be described about the four triangles of the figure which have these points for their vertices: then these circles all meet in one point, and their centres will be in one circumference.

18. A tower of given height is built on the side of a hill whose angle of inclination to the horizon is given; and a flagstaff of given length surmounts the tower: where must the spectator stand (the height of his eye above the ground also given) to see the flagstaff under a given angle, and under the greatest angle possible?

19. What will be the path traced out by the middle of a given line which so moves that its extremities may always be in two lines at right angles?

20. Find the point within a given triangle such that lines drawn from it to the angular points may be:—

(*a.*) Equally inclined to each other;
(*b.*) Together the least possible;
(*c.*) The bisectors of the three angles;
(*d.*) The bisectors of the three sides.

21. Produce a given line, so that:—

(*a.*) The rectangle under the two segments, estimated from the point found, may be equal to a given square;

(*b.*) The rectangle under the given line and the part produced may be equal to a given square;

(*c.*) The rectangle under the given line and the whole line when produced shall be equal to a given square.

22. If two circles cut one another, and diameters be drawn through one of the points of section; the other extremities of these diameters and the other point of section will be in a straight line.

23. The opposite sides of a quadrilateral circumscribed to a circle are together equal to the two other opposite sides.

24. If a quadrilateral inscribed in a circle have its pairs of opposite sides produced to meet; the square of the line joining the points of section will be equal to the sum of the squares of the tangents drawn from those points to the circle.

25. If perpendiculars be drawn from any point in the circumference of a circle upon the sides of any triangle inscribed in that circle, then the three points of intersection will be in one line.

Is this true when, instead of perpendiculars, lines making any equal angles with them are drawn?

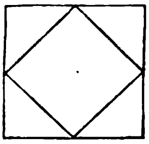
26. If the three perpendiculars in the last proposition be produced to cut the circle, and the points of intersection be joined, the triangle formed by the lines of junction will be equal to the original triangle, whatever point be taken in the circumference from which to draw the perpendiculars.

27. Let a circle ARK and a straight line BP be given, and the diameter AK be perpendicular to BP; and if any circle PRS be described to touch the circle and line, and from the alternate extremity of the diameter a tangent AS be drawn to the latter circle: then whatever be the circle PRS, the tangent AS is of the same length.

BOOK IV.

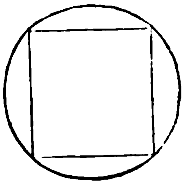
DEFINITIONS.

1. A RECTILINEAL figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

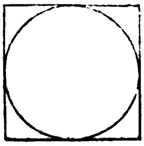


2. In like manner, a figure is said to be described about another figure when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

3. A rectilineal figure is said to be inscribed in a circle when all the angles of the inscribed figure are upon the circumference of the circle.

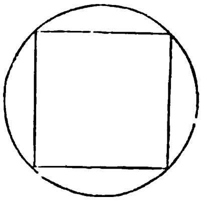


4. A rectilineal figure is said to be described about a circle when each side of the circumscribed figure touches the circumference of the circle.



5. In like manner, a circle is said to be inscribed in a rectilineal figure when the circumference of the circle touches each side of the figure.

6. A circle is said to be described about a rectilineal figure when the circumference of the circle passes through all the angular points of the figure about which it is described.



7. A straight line is said to be placed in a circle when the extremities of it are in the circumference of the circle.

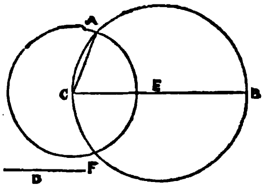
PROPOSITIONS.

PROPOSITION I.

Prob. *In a given circle, to place a straight line equal to a given straight line not greater than the diameter of the circle.*

Let ABC be the given circle, and D the given straight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then if BC is equal to D, the thing required is done; for in the circle ABC, a straight line BC is placed equal to D:



But if it is not, BC is greater than D: make CE equal (3. I.) to D, and from the centre C, at the distance CE, describe the circle AEF, and join CA:

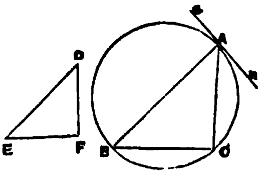
Therefore, because C is the centre of the circle AEF, CA is equal to CE: but D is equal to CE; therefore D is equal to CA. Wherefore, in the circle ABC, a straight line is placed equal to the given straight line D, which is not greater than the diameter of the circle. Which was to be done.

PROPOSITION II.

Prob. *In a given circle to inscribe a triangle equiangular to a given triangle.*

Let ABC be the given circle, and DEF the given triangle; it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw (17. III.) the straight line GAH touching the circle in the point A; and at the point A, in the straight line AH, make (23. I.) the angle HAC equal to the angle DEF; and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC:



Therefore, because HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal (32. III.) to the angle ABC in the alternate segment of the circle:

But HAC is equal to the angle DEF; therefore also the angle ABC is equal to DEF:

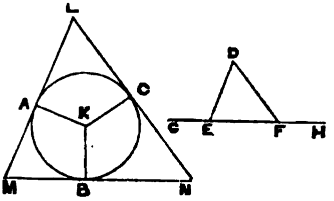
For the same reason, the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal (32. I.) to the remaining angle EDF: wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC. Which was to be done.

PROPOSITION III.

Prob. *About a given circle to describe a triangle equiangular to a given triangle.*

Let ABC be the given circle, and DEF the given triangle; it is required to describe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points G, H, and find (1. III.) the centre K of the circle ABC, and from it draw any straight line KB; at the point K in the straight line KB, make (23. I.) the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C, draw the straight lines LAM, MBN, NCL, touching (17. III.) the circle ABC:



Therefore, because LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB, KC the angles at the points A, B, C are right (18. III.) angles.

And because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles, and that two of them KAM, KBM are right angles, the other two AKB, AMB are equal to two right angles:

But the angles DEG, DEF are likewise equal (13. I.) to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF.

In like manner, the angle LNM may be demonstrated to be equal to DFE; and therefore the remaining angle MLN is equal (32. I.) to the remaining angle EDF: wherefore the triangle LMN is equiangular to the triangle DEF; and it is described about the circle ABC. Which was to be done.

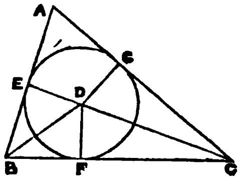
PROPOSITION IV.

Prob. *To inscribe a circle in a given triangle.*

Let the given triangle be ABC; it is required to inscribe a circle in ABC.

Bisect (9. I.) the angles ABC, BCA by the straight lines BD, CD, meeting one another in the point D, from which draw (12. I.) DE, DF, DG perpendiculars to AB, BC, CA:

And because the angle EBD is equal to the angle FBD, for the angle ABC is bisected by BD (*Constr.*), and that the right angle BED is equal to the right angle BFD; the two triangles



EBD, FBD have two angles of the one equal to two angles of the other; and the side BD, which is opposite to one of the equal angles in each, is common to both; therefore their other sides shall be equal (26. I.); wherefore DE is equal to DF:

For the same reason, DG is equal to DF;

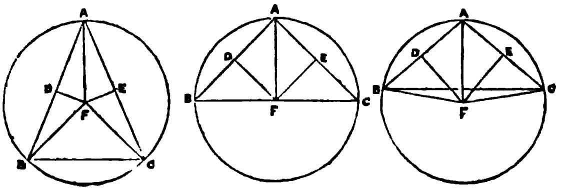
Therefore the three straight lines DE, DF, DG are equal to one another, and the circle described from the centre D, at the distance of any of them, shall pass through the extremities of the other two, and touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it touches (16. III.) the circle; therefore the straight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inscribed in the triangle ABC. Which was to be done.

PROPOSITION V.

Prob. *To describe a circle about a given triangle.*

Let the given triangle be ABC; it is required to describe a circle about ABC.

Bisect (10. I.) AB, AC, in the points D, E, and from these points draw DF, EF at right angles (11. I.) to AB, AC; DF, EF produced, meet one another: for if they do not meet, they are parallel, wherefore AB, AC, which are at right angles to them, are parallel; which is absurd: let them meet in F, and join FA; also if the point F be not in BC, join BF, CF:



Then, because AD is equal to DB, and DF common, and at right angles to AB, the base AF is equal (4. I.) to the base FB.

In like manner, it may be shown that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC are equal to one another: wherefore the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other two, and be described about the triangle ABC. Which was to be done.

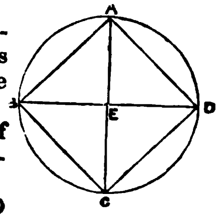
Cor. And it is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but, when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and, if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle: wherefore, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROPOSITION VI.

PROB. *To inscribe a square in a given circle.*

Let ABCD be the given circle; it is required to inscribe a square in ABCD.

Draw the diameters AC, BD at right angles to one another, and join AB, BC, CD, DA.



Because BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the base BA is equal (4. I.) to the base AD:

And, for the same reason, BC, CD are each of them equal to BA, or AD; therefore the quadrilateral figure ABCD is equilateral.

It is also rectangular; for the straight line BD being the diameter of the circle ABCD, BAD is a semicircle; wherefore the angle BAD is a right (31. III.) angle:

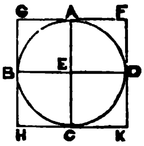
For the same reason, each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular: and it has been shown to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD. Which was to be done.

PROPOSITION VII.

PROB. *To describe a square about a given circle.*

Let ABCD be the given circle; it is required to describe a square about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another; and through the points A, B, C, D, draw (17. III.) FG, GH, HK, KF touching the circle:



And because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right (18. III.) angles; for the same reason the angles at the points B, C, D are right angles:

And because the angle AEB is a right angle, as likewise is EBG; GH is parallel (28. I.) to AC:

For the same reason AC is parallel to FK: and in like manner GF, HK may each of them be demonstrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF is therefore equal (34. I.) to HK, and GH to FK:

And because AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK; GH, FK are each of them equal to GF, or HK; therefore the quadrilateral figure FGHK is equilateral.

It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB (34. I.) is likewise a right angle:

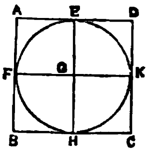
In the same manner, it may be shown that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular: and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD. Which was to be done.

PROPOSITION VIII.

Prob. *To inscribe a circle in a given square.*

Let ABCD be the given square; it is required to inscribe a circle in ABCD.

Bisect (10. I.) each of the sides AB, AD in the points F, E, and through E draw (31. I.) EH parallel to AB or DC, and through F draw FK parallel to AD or BC;



Therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram; and their opposite sides are equal (34. I.): and because AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the sides opposite to these are equal, viz., FG to GE:

In the same manner it may be demonstrated that GH, GK are each of them equal to FG or GE; therefore the four straight lines GE, GF, GH, GK are equal to one another; and the circle described from the centre G at the distance of one of them, shall pass through the extremities of the other three, and touch the straight lines AB, BC, CD, DA; because the angles at the points, E, F, H, K, are right (29. I.) angles, and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle (16. III.); therefore each of the straight lines AB, BC, CD, DA, touches the circle, which therefore is inscribed in the square ABCD. Which was to be done.

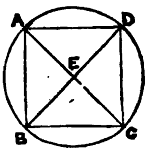
PROPOSITION IX.

Prob. *To describe a circle about a given square.*

Let ABCD be the given square; it is required to describe a circle about it.

Join AC, BD, cutting one another in E:

And because DA is equal to AB, and AC common to the triangles DAC, BAC, the two sides DA, AC are equal to the two BA, AC, each to each; and the base DC is equal to the base BC; wherefore the angle DAC is equal (8. I.) to the angle BAC, and the angle DAB is bisected by the straight line AC:



In the same manner, it may be demonstrated that the angles ABC, BCD, CDA are severally bisected by the straight lines BD, AC;

Therefore, because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC, the angle EAB is equal to the angle EBA; wherefore the side EA is equal (6. I.) to the side EB:

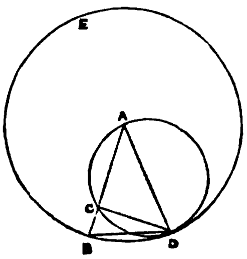
In the same manner, it may be demonstrated that the straight lines EC, ED are each of them equal to EA, or EB; therefore the four straight lines EA, EB, EC, ED are equal to one another; and the circle described from the centre E, at the distance of one of them, shall pass through the extremities of the other three, and be described about the square ABCD. Which was to be done.

PROPOSITION X.

Prob. *To describe an isosceles triangle, having each of the angles at the base double of the third angle.*

Take any straight line AB, and divide (11. II.) it in the point C, so that the rectangle AB, BC be equal to the square of CA; and from the centre A, at the distance AB, describe the circle BDE, in which place (1. IV.) the straight line BD equal to AC, which is not greater than the diameter of the circle BDE; join DA, DC, and about the triangle ADC describe (5. IV.) the circle ACD: the triangle ABD is such as is required, that is, each of the angles ABD, ADB is double of the angle BAD.

Because the rectangle AB, BC is equal to the square of AC, and that AC is equal to BD, the rectangle AB, BC is equal to the square of BD:



And because from the point B, without the circle ACD, two straight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of BD which meets it, the straight line BD touches (37. III.) the circle ACD:

And because BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal (32. III.) to the angle DAC in the alternate segment of the circle:

To each of these add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA, DAC:

But the exterior angle BCD is equal (32. I.) to the angles CDA, DAC; therefore also BDA is equal to BCD:

But BDA is equal (5. I.) to the angle CBD, because the side AD is equal to the side AB; therefore CBD, or DBA, is equal to BCD; and consequently the three angles BDA, DBA, BCD are equal to one another:

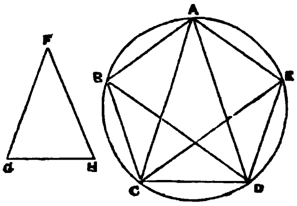
And because the angle DBC is equal to the angle BCD, the side BD is equal (6. I.) to the side DC: but BD was made equal to CA, therefore also CA is equal to CD, and the angle CDA equal (5. I.) to the angle DAC; therefore the angles CDA, DAC together, are double of the angle DAC: but BCD is equal (32. I.) to the angles CDA, DAC; therefore also BCD is double of DAC, and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB. Wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle. Which was to be done.

PROPOSITION XI.

Prob. *To inscribe an equilateral and equiangular pentagon in a given circle.*

Let ABCDE be the given circle; it is required to inscribe an equilateral and equiangular pentagon in the circle ABCDE.

Describe (10. IV.) an isosceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle ABCDE, inscribe (2. IV.) the triangle ACD, equiangular to the triangle FGH, so that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA, equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. Bisect (9. I.) the angles ACD, CDA by the straight lines CE, DB; and join AB, BC, DE, EA: ABCDE is the pentagon required.



Because each of the angles ACD, CDA is double of CAD, and are bisected by the straight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another:

But equal angles stand upon equal (26. III.) circumferences; therefore the five circumferences, AB, BC, CD, DE, EA are equal to one another.

And equal circumferences are subtended by equal (29. III.) straight lines; therefore the five straight lines AB, BC, CD, DE, EA are equal to one another: wherefore the pentagon ABCDE is equilateral. It is also equiangular; because the circumference AB is equal to the circumference DE, if to each be added BCD, the whole ABCD is equal to the whole EDCB:

And the angle AED stands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefore the angle BAE is equal (27. III) to the angle AED:

For the same reason, each of the angles ABC, BCD, CDE, is equal to the angle BAE, or AED; therefore the pentagon ABCDE is equiangular; and it has been shown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed. Which was to be done.

PROPOSITION XII.

PROB. *To describe an equilateral and equiangular pentagon about a given circle.*

Let ABCDE be the given circle; it is required to describe an equilateral and equiangular pentagon about the circle ABCDE.

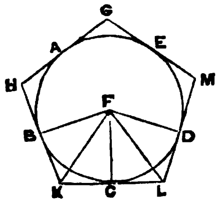
Let the angles of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, E, so that the circumferences AB, BC, CD, DE, EA are equal (11. IV.); and through the points A, B, C, D, E, draw GH, HK, KL, LM, MG, touching (17. 3) the circle; take the centre F, and join FB, FK, FC, FL, FD:

And because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular (18. III.) to KL, therefore each of the angles at C is a right angle: for the same reason, the angles at the points B, D are right angles:

And because FCK is a right angle, the square of FK is equal (47. I.) to the squares of FC, CK: for the same reason, the square of FK is equal to the squares of FB, BK; therefore the squares of FC, CK are equal to the squares of FB, BK; of which the square of FC is equal to the square of FB; the remaining square of CK is therefore equal to the remaining square of BK, and the straight line CK equal to BK:

And because FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK, each to each; and the base BK is equal to the base KC; therefore the angle BFK is equal (8. I.) to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC:

For the same reason, the angle CFD is double of the angle CFL, and CLD double of CLF:



And because the circumference BC is equal to the circumference CD, the angle BFC is equal (27. III.) to the angle CFD; and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL:

And the right angle FCK is equal to the right angle FCL; therefore, in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each; and the side FC, which is adjacent to the equal angles in each, is common to both; therefore the other sides shall be equal (26. I.) to the other sides, and the third angle to the third angle; therefore the straight line KC is equal to CL, and the angle FKC to the angle FLC:

And because KC is equal to CL, KL is double of KC. In the same manner it may be shown that HK is double of BK: and because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK, HK shall be equal to KL:

In like manner it may be shown that GH, GM, ML are each of them equal to HK, or KL; therefore the pentagon GHKLM is equilateral.

It is also equiangular; for since the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonstrated, the angle HKL is equal to KLM:

And in like manner it may be shown that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM; therefore the the five angles GHK, HKL, KLM, LMG, MGH, being equal to one another, the pentagon GHKLM is equiangular: and it is equilateral, as was demonstrated; and it is described about the circle ABCDE. Which was to be done.

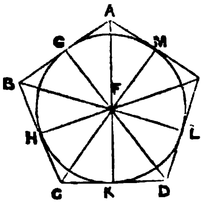
PROPOSITION XIII.

PROB. *To inscribe a circle in a given equilateral and equiangular pentagon.*

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inscribe a circle in the pentagon ABCDE.

Bisect (9. I.) the angles BCD, CDE by the straight lines CF, DF, and from the point F, in which they meet, draw the straight lines, FB, FA, FE:

Therefore, since BC is equal to CD, and CF common to the triangles BCF, DCF, the two sides BC, CF are equal to the two, DC, CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal (4. I.) to the base FD, and the other angles to the other angles, to which the equal sides are opposite; therefore the angle CBF is equal to the angle CDF:



And because the angle CDE is double of CDF and that CDE is equal to CBA, and CDF to CBF: CBA is also double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is bisected by the straight line BF:

In the same manner it may be demonstrated, that the angles BAE, AED are bisected by the straight lines AF, FE.

From the point F, draw (12. I.) FG, FH, FK, FL, FM perpendiculars to the straight lines AB, BC, CD, DE, EA:

And because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC, in the triangles FHC, FKC there are two angles of one equal to two angles of the other; and the side FC, which is opposite to one of the equal angles in each, is common to both; therefore the other sides shall be equal (26. I.), each to each; wherefore the perpendicular FH is equal to the perpendicular FK:

In the same manner it may be demonstrated that FL, FM, FG are each of them equal to FH, or FK;

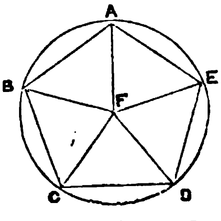
Therefore the five straight lines FG, FH, FK, FL, FM are equal to one another; wherefore the circle described from the centre F, at the distance of one of these five, shall pass through the extremities of the other four, and touch the straight lines AB, BC, CD, DE, EA, because the angles at the points G, H, K, L, M are right angles, and that a straight line drawn from the extremity of the diameter of a circle at right angles to it touches (16. III.) the circle; therefore each of the straight lines AB, BC, CD, DE, EA touches the circle: wherefore it is inscribed in the pentagon ABCDE. Which was to be done.

PROPOSITION XIV.

PROB. *To describe a circle about a given equilateral and equiangular pentagon.*

Let ABCDE be the given equilateral and equiangular pentagon; it is required to describe a circle about it.

Bisect (9. I.) the angles BCD, CDE by the straight lines CF, FD, and from the point F, in which they meet, draw the straight lines FB, FA, FE, to the points B, A, E.



It may be demonstrated, in the same manner as in the preceding proposition, that the angles CBA, BAE, AED are bisected by the straight lines FB, FA, FE:

And because the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the side CF is equal (6. I.) to the side FD:

In like manner it may be demonstrated that FB, FA, FE are each of them equal to FC or FD; therefore the five straight lines FA, FB, FC, FD, FE are equal to one another; and the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon ABCDE. Which was to be done.

PROPOSITION XV.

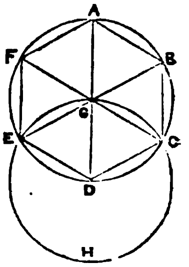
PROB. *To inscribe an equilateral and equiangular hexagon in a given circle.*

Let ABCDEF be the given circle; it is required to inscribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the distance DG, describe the circle EGCH; join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA: the hexagon ABCDEF is equilateral and equiangular.

Because G is the centre of the circle ABCDEF, GE is equal to GD:

And because D is the centre of the circle EGCH, DE is equal to DG;



Wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another, because the angles at the base of an isosceles triangle are equal (*Cor.* 5. I.); and the three angles of a triangle are equal (32. I.) to two right angles; therefore the angle EGD is the third part of two right angles:

In the same manner it may be demonstrated, that the angle DGC is also the third part of two right angles:

And because the straight line GC makes with EB the adjacent angles EGC, CGB equal (13. I.) to two right angles, the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB are equal to one another; and to these are equal (15. I.) the vertical opposite angles BGA, AGF, FGE; therefore the six angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another:

But equal angles stand upon equal (26. III) circumferences; therefore the six circumferences AB, BC, CD, DE, EF, FA are equal to one another:

And equal circumferences are subtended by equal (29. III.) straight lines; therefore the six straight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is also equiangular; for, since the circumference AF is equal to ED, to each of these add the circumference ABCD; therefore the whole circumference FABCD shall be equal to the whole EDCBA: and the angle FED stands upon the circumference FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal (27. III.) to FED: in the same manner it may be demonstrated, that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED; therefore the hexagon is equiangular; and it is equilateral, as was shown; and it is inscribed in the given circle ABCDEF. Which was to be done.

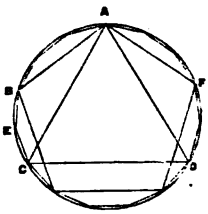
Cor. From this it is manifest, that the side of the hexagon is equal to the straight line from the centre, that is, to the semidiameter of the circle.

And if through the points A, B, C, D, E, F, there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon: and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

PROPOSITION XVI.

Prob. *To inscribe an equilateral and equiangular quindecagon in a given circle.*

Let ABCD be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle ABCD.



Let AC be the side of an equilateral triangle inscribed (2. iv.) in the circle, and AB the side of an equilateral and equiangular pentagon inscribed (11. iv.) in the same;

Therefore, of such equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five;

And the circumference AB, which in the fifth part of the whole, contains three;

Therefore BC, their difference, contains two of the same parts:

Bisect (30. iii.) BC in E: therefore BE, EC are each of them the fifteenth part of the whole circumference ABCD; therefore, if the straight lines BE, EC be drawn and straight lines equal to them be placed round (1. iv.) in the whole circle, an equilateral and equiangular quindecagon shall be inscribed in it. Which was to be done.

And in the same manner as was done in the pentagon, if, through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it: and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

Exercises on Book IV.

1. In a given circle to inscribe a line of given length, so as:—
   - (*a*) To pass through a given point;
   - (*b*) To touch a given circle;
   - (*c*) To make a given angle with a given line;
   - (*d*) To cut off one-third of another given circumference.

In all cases the limits of possibility to be assigned.

2. If all possible triangles having the same base and equal vertical angles be supposed to exist, show that the centres of the inscribed and escribed circles are situated upon the circumferences of determinable circles.

3. Describe three circles, each touching the other two, and the three centres being three given points, in all possible ways.

4. Describe a circle which shall pass through a given point and touch a given line, and a given circle; and assign the limits of possibility.

5. From the three angles of a triangle draw perpendiculars to the opposite sides; and join the intersections two and two: then

(*a*) The perpendiculars intersect in the same point;

(*b*) They bisect the angles of the second triangle;

(*c*) The circle about the second triangle bisects the sides of the first;

(*d*) The diameter of this circle is half that described about the first triangle.

6. In the figure to IV. 10., produce DC to meet the circle in F, and show that:—

(*a*) AC is the side of a regular pentagon inscribed in the smaller circle ACD;

(*b*) Of a regular decagon, in the larger one BED;

(*c*) The angle ABF will be triple of BFD.

7. If there be drawn lines to the alternate angles of a regular pentagon, their intersections will give the angular points of another regular pentagon; and if the alternate sides be produced, their intersections will be the angular points of a third regular pentagon; and the circles circumscribing and inscribed in these pentagons will all be concentric.

8. Divide a right angle into three, four, five, six, eight, ten, twelve and fifteen equal parts.

9. Every regular polygon may have a circle inscribed within it, and another circumscribed about it; and these circles will be concentric.

10. An equilateral figure inscribed in a circle, or circumscribed about it, is equiangular; and an equiangular figure so inscribed or circumscribed is equilateral.

11. A triangle being constituted of the sides of the regular pentagon, hexagon and decagon inscribed in the same circle, will be right angled; and the same is true of the regular inscribed triangle, square, and hexagon.

12. Given the radius of the inscribed circle and the perimeter of the triangle, to construct it in the cases where the third datum is:—

(*a*) The sum of the sides;

(*b*) The difference of the sides;

(*c*) The perpendicular on the base;

(*d*) The radius of the circumscribing circle;

(*e*) The vertical angle;

(*f*) The difference of the angles at the base.

13. If there be equilateral triangles described externally on the three sides of any triangle, the three distances between the vertices of these and the opposite angles of the original triangle will be all equal:

And if the original triangle be right-angled, two of the equilateral triangles will together be equal to the third.

14. A regular polygon and a circle concentric with it being given, and from any point in the circle draw lines to all the angles, and likewise lines perpendicular to the sides: the sum of the perpendiculars is constant; and the sum of the squares on all the other lines is constant, wherever in the circumference the point be taken.

BOOK V.

DEFINITIONS.

1. A LESS magnitude is said to be a part of a greater magnitude, when the less measures the greater; that is, 'when the less is contained a certain number of times exactly in the greater.'

2. A greater magnitude is said to be a multiple of a less, when the greater is measured by the less; that is, 'when the greater contains the less a certain number of times exactly.'

3. 'Ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.'

4. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

5. The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

6. Magnitudes which have the same ratio are called proportionals. 'N.B. When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third to the fourth.'

7. When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

8. 'Analogy, or proportion, is the similitude of ratios.'

9. Proportion consists of three terms at least.

10. When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

11. When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, etc., increasing the denomination still by unity, in any number of proportionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and the ratio of B to C, and the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that G has to H; and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

12. In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

Geometers make use of the following technical words, to 'signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals.'

13. Permutando, or alternando, by permutation or alternately. This word is used when there are four proportionals, and it is inferred, that the first has the same ratio to the third which the second has to the fourth; or, that the first is to the third as the second to the fourth; as is shown in the 16th Prop. of this 5th Book.

14. Invertendo, by inversion; when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third. Prop. B., Book V.

15. Componendo, by composition; when there are four proportionals, and it is inferred that the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth. 18th Prop., Book V.

16. Dividendo, by division; when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. 17th Prop., Book V.

17. Convertendo, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E., Book V.

18. Ex æquali (sc. distantia), or ex æquo, from equality of distance; when there is any number of magnitudes more than two, and as many others, such that they are proportionals, when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others. 'Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two.'

19. Ex. æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in the 22nd Prop., Book V.

20. Ex. æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion.[*] This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order; and the inference is as in the 18th Definition. It is demonstrated in the 23rd Prop. of Book V.

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

2. Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

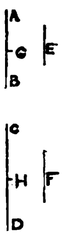
PROPOSITIONS.

PROPOSITION I.

THEOR. *If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.*

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each: whatsoever multiple AB is of E, the same multiple shall AB and CD together, be of E and F together.

Because AB is the same multiple of E that CD is of F, as many magnitudes as are in AB equal to E, so many are there in CD equal to F. Divide AB into magnitudes equal to E, viz., AG, GB; and CD into CH, HD, equal each of them to F: the number therefore of the magnitudes CH, HD, shall be equal to the number of the others AG, GB:



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[*] Prop. 4., Lib. II., Archimedis de Sphærâ et Cylindro.

And because AG is equal to E, and CH to F, therefore AG and CH together are equal to (*Ax.* 2. I.) E and F together.

For the same reason, because GB is equal to E, and HD to F; GB and HD together are equal to E and F together.

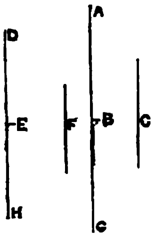
Wherefore, as many magnitudes as are in AB equal to E, so many are there in AB, CD together, equal to E and F together: therefore, whatsoever multiple AB is of E, the same multiple is AB and CD together of E and F together.

Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each; whatsoever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the others: 'For the same demonstration holds in any number of magnitudes which was here applied to two.' Q. E. D.

PROPOSITION II.

THEOR. *If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second that the third together with the sixth is of the fourth.*

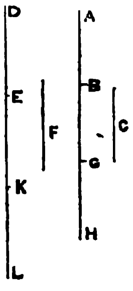
Let AB the first be the same multiple of C the second that DE the third is of F the fourth; and BG the fifth the same multiple of C the second, that EH the sixth is of F the fourth: then is AG, the first together with the fifth, the same multiple of C the second that DH, the third together with the sixth, is of F the fourth.



Because AB is the same multiple of C that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F;

In like manner, as many as there are in BG equal to C, so many are there in EH equal to F:

As many, then, as are in the whole AG equal to C, so many are there in the whole DH equal to F; therefore AG is the same multiple of C that DH is of F; that is, AG, the first and fifth together, is the same multiple of the second C, that DH the third and sixth together is of the fourth F. If, therefore, the first be the same multiple, etc. Q. E. D.

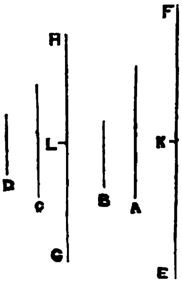


COR. 'From this it is plain, that if any number of magnitudes AB, BG, GH, be multiples of another C; and as many DE, EK, KL, be the same multiples of F, each of each; the whole of the first, viz., AH, is the same multiple of C, that the whole of the last, viz. DL, is of F.'

PROPOSITION III.

Theor. *If the first be the same multiple of the second which the third is of the fourth, and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.*

Let A the first be the same multiple of B the second, that C the third is of D the fourth; and of A, C, let the equimultiples EF, GH be taken: then EF is the same multiple of B, that GH is of D.



Because EF is the same multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as are in GH equal to C; let EF be divided into the magnitudes EK, KF, each equal to A; and GH into GL, LH, each equal to C; the number therefore of the magnitudes EK, KF, shall be equal to the number of the others, GL, LH;

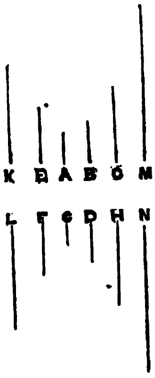
And because A is the same multiple of B, that C is of D, and that EK is equal to A, and GL to C; therefore EK is the same multiple of B, that GL is of D:

For the same reason, KF is the same multiple of B, that LH is of D; and so if there be more parts in EF, GH, equal to A, C:

Because, therefore, the first EK is the same multiple of the second B, which the third GL is of the fourth D, and that the fifth KF is the same multiple of the second B, which the sixth LH is of the fourth D; EF the first together with the fifth, is the same multiple (2. v.) of the second B, which GH the third together with the sixth, is of the fourth D. If, therefore, the first, etc. Q. E. D.

PROPOSITION IV.

Theor. *If the first of four magnitudes has the same ratio to the second which the third has to the fourth, then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; viz. 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'*



Let A the first have to B the second, the same ratio which the third C has to the fourth D; and of A and C, let there be taken any equimultiples whatever E, F; and of B and D, any equimultiples whatever G, H; then E has the same ratio to G, which F has to H.

Take of E and F, any equimultiples whatever K, L, and of G, H, any equimultiples whatever M, N.

Then, because E is the same multiple of A, that F is of C; and of E and F have been taken equimultiples K, L; therefore K is the same multiple of A, that L is of C (3. v.).

For the same reason, M is the same multiple of B, that N is of D.

And because, as A is to B, so is C to D (*Hyp.*), and of A and C have been taken certain equimultiples K, L; and of B and D have been taken certain equimultiples M, N; if therefore K be greater than M, L is greater than N; and if equal, equal; if less, less (5 *Def.* v.). And K, L are any equimultiples (*Constr.*) whatever of E, F; and M, N any whatever of G, H: as therefore E is to G, so is (5 *Def.* v.) F to H. Therefore, if the first, etc. Q. E. D.

COR. Likewise, if the first has the same ratio to the second which the third has to the fourth, then also any equimultiples whatever of the first and third have the same ratio to the second and fourth: and in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

Let A the first have to B the second, the same ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; then E is to B, as F to D.

Take of E, F, any equimultiples whatever K, L, and of B, D, any equimultiples whatever G, H; then it may be demonstrated, as before, that K is the same multiple of A, that L is of C:

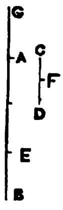
And because (*Hyp.*) A is to B, as C is to D, and of A and C certain equimultiples have been taken, viz. K and L; and of B and D, certain equimultiples G, H; therefore, if K be greater than G, L is greater than H; and if equal, equal; if less, less (5 *Def.* v.).

And K, L are any (*Constr.*) equimultiples of E, F, and G, H any whatever of B, D; as therefore E is to B (5 *Def.* v.), so is F to D. And in the same way the other case is demonstrated.

PROPOSITION V.

THEOR. *If one magnitude be the same multiple of another which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.*

Let the magnitude AB be the same multiple of CD, that AE taken from the first is of CF taken from the other; the remainder EB shall be the same multiple of the remainder FD, that the whole AB is of the whole CD.



Take AG the same multiple of FD, that AE is of CF; therefore AE is (1. v.) the same multiple of CF, that EG is of CD:

But AE, by the hypothesis, is the same multiple of CF, that AB is of CD; therefore EG is the same multiple of CD that AB is of CD; wherefore EG is equal to AB (1 *Ax.* v.):

Take from them the common magnitude AE, and the remainder AG is equal to the remainder EB.

Wherefore, since AE is the same multiple of CF, that AG is of FD, and that AG is equal to EB, therefore AE is the same multiple of CF, that EB is of FD: but AE is the same multiple of CF (*Hyp.*) that AB is of CD; therefore EB is the same multiple of FD, that AB is of CD. Therefore, if one magnitude, etc. Q. E. D.

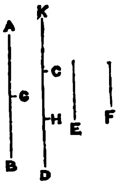
PROPOSITION VI.

THEOR. *If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders are either equal to these others or equimultiples of them.*

Let the two magnitudes AB, CD be equimultiples of the two E, F, and AG, CH taken from the first two be equimultiples of the same E, F; the remainders GB, HD are either equal to E, F, or equimultiples of them.

First, let GB be equal to E; HD is F qeut. Make CK equal to F:

And because AG is the same multiple of E, that CH is of F, and that GB is equal to E, and CK to F; therefore AB is the same multiple of E, that KH is of F:

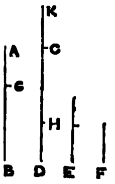


But AB, by the hypothesis, is the same multiple of E, that CD is of F; therefore KH is the same multiple of F, that CD is of F; wherefore KH is equal to CD (1 *Ax.* v.):

Take away the common magnitude CH, then the remainder KC is equal to the remainder HD: but KC is equal (*Constr.*) to F; HD therefore is equal to F.

But let GB be a multiple of E; then HD is the same multiple of F: make CK the same multiple of F, that GB is of E:

And because AG is the same multiple of E, that CH is of F; and GB the same multiple of E, that CK is of F; therefore AB is the same multiple of E (*Hyp.*), that KH is of F (2. v.):



But AB is the same multiple of E (*Hyp.*), that CD is of F; therefore KH is the same multiple of F, that CD is of it; wherefore KH is equal (1 *Ax.* v.) to CD:

Take away CH from both; therefore the remainder KC is equal to the remainder HD:

And because GB is the same multiple of E, that KC is of F, and that KC is equal to HD; therefore HD is the same multiple of F, that GB is of E. If, therefore, two magnitudes, etc. Q. E. D.

PROPOSITION A.

THEOR. *If the first of four magnitudes has to the second the same ratio which the third has to the fourth, then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.*

Take any equimultiples of each of them, as the doubles of each; then, by Def. 5th of this book, if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth: but if the first be greater than the second, the double of the first is greater than the double of the second; wherefore also the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth. In like manner, if the first be equal to the second, or less than it, the third can be proved to be equal to the fourth, or less than it. Therefore, if the first, etc. Q. E. D.

PROPOSITION B.

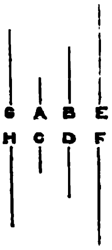
Theor. *If four magnitudes are proportionals, they are proportionals also when taken inversely.*

If the magnitude A be to B, as C is to D, then also inversely B is to A, as D to C.

Take of B and D, any equimultiples whatever E and F; and of A and C, any equimultiples whatever G and H.

First, let E be greater than G, then G is less than E: and because (*Hyp.*) A is to B as C is to D; and of A and C, the first and third, G and H are equimultiples; and of B and D, the second and fourth, E and F are equimultiples; and that G is less than E, H is also (5 *Def.* v.) less than F; that is, F is greater than H; if therefore E be greater than G, F is greater than H:

In like manner, if E be equal to G, F may be shown to be equal to H; and if less, less; and E, F are any equimultiples (*Constr.*) whatever of B and D, and G, H any whatever of A and C; therefore (5 *Def.* v.), as B is to A, so is D to C. If, then, four magnitudes, etc. Q. E. D.



PROPOSITION C.

Theor. *If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second as the third is to the fourth.*

Let the first A be the same multiple of B the second, that C the third is of the fourth D: A is to B as C is to D.

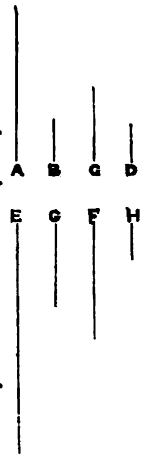
Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H:

Then, because, A is the same (*Hyp.*) multiple of B that C is of D; and that E is the same (*Constr.*) multiple of A that F is of C; E is the same multiple of B that F is of D (3. v.), therefore E and F are the same multiples of B and D:

But G and H are equimultiples (*Constr.*) of B and D; therefore, if E be a greater multiple of B than G is, F is a greater multiple of D than H is of D; that is, if E be greater than G, F is greater than H:

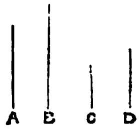
In like manner, if E be equal to G, or less, F is equal to H, or less than it:

But E, F are any equimultiples (*Constr.*) whatever of A, C; and G, H any equimultiples whatever of B, D; therefore A is to B, as C is to D (5 *Def.* v.).



Next, let the first A be the same part of the second B, that the third C is of the fourth D: A is to B, as C is to D:

For B is the same multiple of A, that D is of C: wherefore, by the preceding case, B is to A, as D is to C; and inversely (*B.* v.) A is to B, as C is to D. Therefore, if the first be the same multiple, etc. Q. E. D.



PROPOSITION D.

Theor. *If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.*

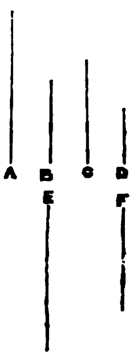
Let A be to B, as C is to D; and first, let A be a multiple of B; C is the same multiple of D.

Take E equal to A; and whatever multiple A or E is of B, make F the same multiple of D:

Then, because (*Hyp.*) A is to B, as C is to D; and of B the second, and D the fourth, equimultiples have been taken, E and F; A is to E, as C to F (*Cor.* 4. v.):

But A is equal (*Constr.*) to E, therefore C is equal to F (*A.* v.).

And F is the same (*Constr.*) multiple of D, that A is of B. Wherefore C is the same multiple of D, that A is of B.



Next, let the first A be a part of the second B; C the third is the same part of the fourth D—(See figure at bottom of Proposition C.)

Because A is to B, as C is to D; then inversely, B is (*B.* v.) to A, as D to C: but A is a part of B, therefore B is a multiple of A; and, by the preceding case, D is the same multiple of C; that is, C is the same part of D, that A is of B. Therefore, if the first, etc. Q. E. D.

PROPOSITION VII.

Theor. *Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.*

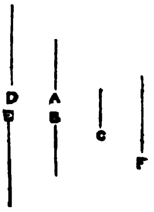
Let A and B be equal magnitudes, and C any other: A and B have each of them the same ratio to C, and C has the same ratio to each of the magnitudes A and B.

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F:

Then, because D is the same (*Constr.*) multiple of A, that E is of B, and that A is equal (*Hyp.*) to B; D is (1 *Ax.* v.) equal to E;

Therefore, if D be greater than F, E is greater than F; and if equal, equal; if less, less;

And D, E are any equimultiples of A, B (*Constr.*), and F is any multiple of C. Therefore (5 *Def.* v.) as A is to C, so is B to C.



Likewise C has the same ratio to A that it has to B.

For, having made the same construction, D may in like manner be shown equal to E: therefore, if F be greater than D, it is likewise greater than E; and if equal, equal; if less, less:

And F is any multiple whatever of C; and D, E are any equimultiples whatever of A, B; therefore, C is to A as C is to B (5 *Def.* v.). Therefore, equal magnitudes, etc. Q. E. D.

PROPOSITION VIII.

THEOR. *Of unequal magnitudes the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.*

Let AB, BC be unequal magnitudes, of which AB is the greater, and let D be any magnitude whatever: AB has a greater ratio to D than BC to D; and D has a greater ratio to BC than unto AB.

If the magnitude which is not the greater of the two AC, CB, be not less than D, take EF, FG, the doubles of AC, CB, as in fig. 1. But if that which is not the greater of the two AC, CB, be less than D (as in figs. 2 and 3), this magnitude can be multiplied, so as to become greater than D, whether it be AC or CB. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the same multiple of CB; therefore EF and FG are each of them greater than D: and in every one of the cases, take H the double of D, K its triple, and so on, till the multiple of D be that which first becomes greater than FG: let L be that multiple of D which is first greater than FG, and K the multiple of D which is next less than L.

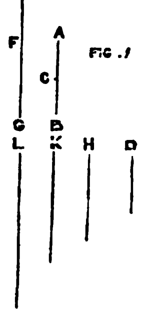


FIG. 1

Then, because L is the multiple of D, which is the first that becomes greater than FG, the next preceding multiple K is not greater than FG; that is, FG is not less than K:

And since EF is the same multiple of AC, that FG is of CB; FG is the same multiple of CB, that EG is of AB (1. v.); wherefore EG and FG are equimultiples of AB and CB:

And it was shown that FG was not less than K, and by the construction, EF is greater than D; therefore the whole EG is greater than K and D together:

But K together with D is equal (*Constr.*) to L; therefore EG is greater than L: but FG is not greater than L; and EG, FG are equimultiples of AB, BC, and L is a (*Constr.*) multiple of D; therefore (7 *Def.* v.) AB has to D a greater ratio than BC has to D.

Also, D has to BC a greater ratio than it has to AB.

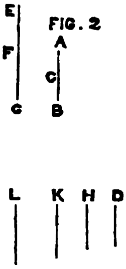


FIG. 2

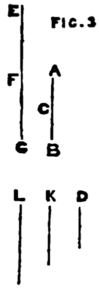


FIG. 3

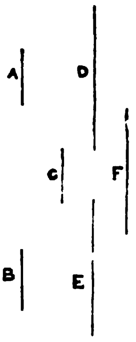
For, having made the same construction, it may be shown, in like manner, that L is greater than FG, but that it is not greater than EG: and L is a multiple of D; and FG, EG are equimultiples of CB, AB; therefore D has to CB a greater ratio (7 *Def.* v.) than it has to AB. Wherefore, of unequal magnitudes, etc. Q. E. D.

PROPOSITION IX.

Theor. *Magnitudes which have the same ratio to the same magnitude, are equal to one another; and those to which the same magnitude has the same ratio, are equal to one another.*

Let A, B have each of them the same ratio to C; A is equal to B.

For, if they are not equal, one of them is greater than the other: let A be the greater; then, by what was shown in the preceding proposition (8. v.), there are some equimultiples of A and B, and some multiple of C, such that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let such multiples be taken; and let D, E be the equimultiples of A, B, and F the multiple of C, so that D may be greater than F, and E not greater than F:



But, because A is to C as (*Hyp.*) B is to C, and of A, B, are taken equimultiples D, E, and of C is taken a multiple F, and that D is greater than F; E shall also be greater than F (5 *Def.* v.): but E is not greater (*Constr.*) than F; which is impossible; A therefore and B are not unequal; that is, they are equal.

Next, let C have the same ratio to each of the magnitudes A and B: A is equal to B.

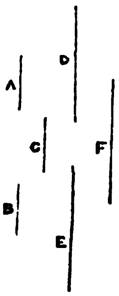
For, if they are not, one of them is greater than the other: let A be the greater; therefore, as was shown in Prop. 8th, there is some multiple F of C, and some equimultiples E and D, of B and A, such, that F is greater than E, and not greater than D: but because C is to B (*Hyp.*), as C is to A, and that F the multiple of the first is greater than E the multiple of the second (5 *Def.* v.); F the multiple of the third is greater than D, the multiple of the fourth: but F is not (*Constr.*) greater than D, which is impossible. Therefore A is equal to B. Wherefore, magnitudes which, etc. Q. E. D.

PROPOSITION X.

Theor. *That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.*

Let A have to C a greater ratio than B has to C; A is greater than B.

For, because A has a greater ratio to C, than B has to C, there are (7 *Def.* v.) some equimultiples of A and B, and some multiple of C, such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it: let them be taken; and let D, E be equimultiples of A, B, and F a multiple of C, such, that D is greater than F, but E is not greater than F; therefore D is greater than E:



And, because D and E are equimultiples of A and B, and D is greater than E; therefore A is (4 *Ax.* v.) greater than B.

Next, let C have a greater ratio to B than it has to A;

B is less than A: for (7 *Def.* v.) there is some multiple F of C, and some equimultiples E and D of B and A, such, that F is greater than E, but is not greater than D; E therefore is less than D:

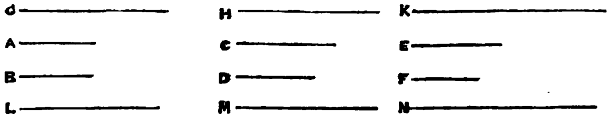
And because E and D are equimultiples of B and A, therefore B is (4 *Ax.* v.) less than A. That magnitude, therefore, etc. Q. E. D.

PROPOSITION XI.

THEOR. *Ratios that are the same to the same ratio, are the same to one another.*

Let A be to B, as C is to D; and as C to D, so let E be to F: A is to B, as E to F.

Take of A, C, E, any equimultiples whatever G, H, K; and of



B, D, F, any equimultiples whatever L, M, N. Therefore, since A is to B as C to D, and G, H are taken equimultiples of A, C, and L, M, of B, D; if G be greater than L, H is greater than M; and if equal, equal; and if less, less. (5 *Def.* v.).

Again, because C is to D, as E is to F, and H, K are taken equimultiples of C, E; and M, N, of D, F; if H be greater than M, K is greater than N; and if equal, equal; and if less, less (5 *Def.* v.).

But if G be greater than L, it has been shown that H is greater than M; and if equal, equal; and if less, less; therefore if G be greater than L, K is greater than N; and if equal, equal; and if less, less:

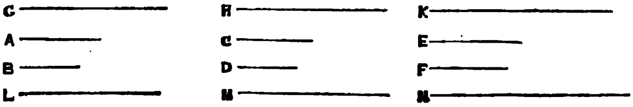
And G, K are any equimultiples whatever of A, E; and L, N any whatever of B, F; therefore, as A is to B, so is E to F (5 *Def.* v.). Wherefore, ratios that, etc. Q. E. D.

PROPOSITION XII.

THEOR. *If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.*

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, as A is to B, so C to D, and E to F: as A is to B, so shall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H, K; and of



B, D, F any equimultiples whatever L, M, N:

Then, because A is to B, as C is to D, and as E to F; and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of

B, D, F; if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if less, less (5 *Def.* v.):

Wherefore, if G be greater than L, then G, H, K together are greater than L, M, N together; and if equal, equal; and if less, less.

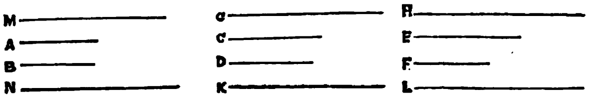
And G, and G, H, K together are any equimultiples of A, and A, C, E together; because if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole (1. v.): for the same reason L, and L, M, N are any equimultiples of B, and B, D, F: as therefore A is to B (5 *Def.* v.) so are A, C, E together to B, D, F together. Wherefore, if any number, etc. Q. E. D.

PROPOSITION XIII.

Theor. *If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth, the first shall also have to the second a greater ratio than the fifth has to the sixth.*

Let A the first have the same ratio to B the second, which C the third has to D the fourth, but C the third, to D the fourth, a greater ratio than E the fifth, to F the sixth: also the first A shall have to the second B a greater ratio than the fifth E to the sixth F.

Because C has a greater ratio to D, than E to F, there are some equimultiples of C and E, and some of D and F, such, that the multiple of C is greater than the multiple of D, but the multiple of E is not greater than the multiple of F (7 *Def.* v.): let such be taken, and of C, E, let G, H be equimultiples, and K, L equimultiples of D, F, such that G be greater than K, but H not greater than L: and whatever multiple G is of C, take M the same multiple of A; and whatever multiple K is of D, take N the same multiple of B:



Then, because A is to B, as C to D, and of A and C, M and G are equimultiples; and of B and D, N and K are equimultiples; if M be greater than N, G is greater than K; and if equal, equal; and if less, less (5 *Def.* v.):

But G is greater (*Constr.*) than K; therefore M is greater than N: but H is not greater (*Constr.*) than L; and M, H are equimultiples of A, E; and N, L equimultiples of B, F; therefore A has a greater ratio to B, than E has to F (7 *Def.* v.). Wherefore, if the first, etc. Q. E. D.

Cor. And if the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second than the fifth has to the sixth.

PROPOSITION XIV.

THEOR. *If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.*

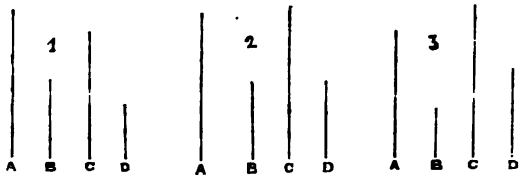
Let the first A have to the second B the same ratio which the third C has to the fourth D: if A be greater than C, B is greater than D.

Because A is greater than C (*Hyp.*), and B is any other magnitude, A has to B a greater ratio than C to B (8. v.):

But, as A is to B, so is C to D; therefore also C has to D a greater ratio than C has to B (13. v.):

But of two magnitudes, that to which the same has the greater ratio is the less (10. v.). Wherefore D is less than B; that is, B is greater than D:

Secondly, if A be equal to C, B is equal to D: for A is to B, as C, that is, A to D; B therefore is equal to D (9. v.).



Thirdly, if A be less than C, B shall be less than D:

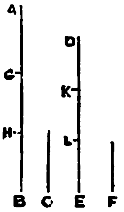
For C is greater than A; and because C is to D, as A is to B, D is greater than B, by the first case; wherefore B is less than D. Therefore, if the first, etc. Q. E. D.

PROPOSITION XV.

THEOR. *Magnitudes have the same ratio to one another which their equimultiples have.*

Let AB be the same multiple of C, that DE is of F: C is to F, as AB to DE.

Because AB is the same multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE: then the number of the first AG, GH, HB, shall be equal to the number of the last, DK, KL, LE:



And because AG, GH, HB are all equal, and that DK, KL, LE are also equal to one another; therefore AG is to DK as GH to KL, and as HB to LE (7. v.):

And as one of the antecedents to its consequent, so are all the antecedents together to all the consequents together (12. v.); wherefore, as AG is to DK, so is AB to DE: but AG is equal to C, and DK to F; therefore, as C is to F, so is AB to DE. Therefore, magnitudes, etc. Q. E. D.

PROPOSITION XVI.

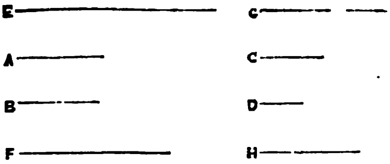
Theor. *If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.*

Let the four magnitudes A, B, C, D be proportionals, viz. as A to B, so C to D: they shall also be proportionals when taken alternately; that is, A is to C, as B to D.

Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever G and H:

And because E is the same multiple of A, that F is of B, and that magnitudes have the same ratio to one another which their equimultiples have (15. v.); therefore A is to B, as E is to F:

But as A is to B, so (*Hyp.*) is C to D; wherefore as C is to D, so (11. v.) is E to F:



Again, because G, H are equimultiples of C, D, as C is to D, so is G to H (15. v.): but as C is to D, so is E to F. Wherefore, as E is to F, so is G to H (11. v.).

But when four magnitudes are proportionals, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; if less, less (14. v.). Wherefore, if E be greater than G, F likewise is greater than H; and if equal, equal; if less, less: and E, F are any (*Constr.*) equimultiples whatever of A, B; and G, H any whatever of C, D: therefore A is to C, as B to D (5 *Def.* v.). If, then, four magnitudes, etc. Q. E. D.

PROPOSITION XVII.

Theor. *If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.*

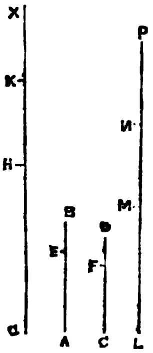
Let AB, BE, CD, DF be the magnitudes, taken jointly, which are proportionals; that is, as AB to BE, so is CD to DF: they shall also be proportionals taken separately; viz. as AE to EB, so CF to FD.

Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN; and again, of EB, FD take any equimultiples whatever KX, NP:

And because GH is the same multiple of AE, that HK is of EB, wherefore GH is the same multiple (1. v.) of AE, that GK is of AB:

But GH is the same multiple of AE, that LM is of CF; wherefore GK is the same multiple of AB, that LM is of CF.

Again, because LM is the same multiple of CF, that MN is of FD; therefore LM is the same multiple (1. v.) of CF, that LN is of CD:



But LM was shown to be the same multiple of CF, that GK is of AB; GK therefore is the same multiple of AB, that LN is of CD; that is, GK, LN are equimultiples of AB, CD.

Next, because HK is the same multiple of EB, that MN is of FD; and that KX is also the same multiple of EB, that NP is of FD; therefore HX is the same multiple (2. v.) of EB, that MP is of FD.

And because AB is to BE, as CD is to DF, and that of AB and CD, GK and LN are equimultiples, and of EB and FD, HX and MP are equimultiples; if GK be greater than HX, then LN is greater than MP; and if equal, equal; and if less, less (5. *Def.* v.):

But if GH be greater than KX, by adding the common part HK to both, GK is greater than HX: wherefore also LN is greater than MP; and by taking away MN from both, LM is greater than NP: therefore, if GH be greater than KX, LM is greater than NP.

In like manner it may be demonstrated, that if GH be equal to KX, LM likewise is equal to NP; and if less, less:

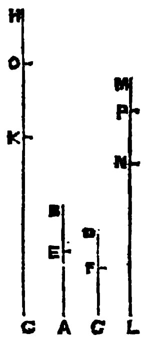
And GH, LM are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD: therefore (5 *Def.* v.), as AE is to EB, so is CF to FD. If, then, magnitudes, etc. Q. E. D.

PROPOSITION XVIII.

THEOR. *If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly; that is, if the first be to the second as the third to the fourth, the first and second together shall be to the second as the third and fourth together to the fourth.*

Let AE, EB, CF, FD be proportionals; that is, as AE to EB, so is CF to FD: they shall also be proportionals when taken jointly; that is, as AB to BE, so CD to DF.

Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN: and again, of BE, DF take any whatever equimultiples KO, NP:



And because KO, NP are equimultiples of BE, DF, and that KH, NM are equimultiples likewise of BE, DF; if KO, the multiple of BE, be greater than KH, which is a multiple of the same BE, NP likewise, the multiple of DF, shall be greater than MN, the multiple of the same DF; and if KO be equal to KH, NP shall be equal to NM; and if less, less.

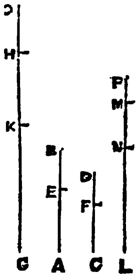
First, let KO be not greater than KH; therefore NP is not greater than NM:

And because GH, HK are equimultiples of AB, BE, and that AB is greater than BE, therefore GH is greater (3 *Ax.* v.) than HK; but KO is not greater than KH, wherefore GH is greater than KO.

In like manner it may be shown, that LM is greater than NP:

Therefore, if KO be not greater than KH, then GH, the multiple of AB, is always greater than KO, the multiple of BE; and likewise LM, the multiple of CD, is greater than NP, the multiple of DF.

Next, let KO be greater than KH; therefore, as has been shown, NP is greater than NM:



And because the whole GH is the same multiple of the whole AB, that HK is of BE, the remainder GK is the same multiple of the remainder AE that GH is of AB (5. v.); which is the same that LM is of CD.

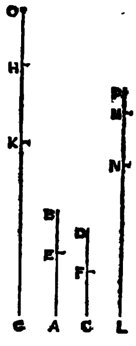
In like manner, because LM is the same multiple of CD, that MN is of DF, the remainder LN is the same multiple of the remainder CF, that the whole LM is of the whole CD (5. v.).

But it was shown that LM is the same multiple of CD, that GK is of AE; therefore GK is the same multiple of AE, that LN is of CF; that is, GK, LN are equimultiples of AE, CF.

And because KO, NP are equimultiples of BE, DF, if from KO, NP there be taken KH, NM, which are likewise equimultiples of BE, DF, the remainders HO, MP are either equal to BE, DF, or equimultiples of them (6. v.).

First, let HO, MP be equal to BE, DF: and because AE is to EB, as CF to FD, and that GK, LN are equimultiples of AE, CF; GK shall be to EB, as LN to FD (*Cor.* 4. v.): but HO is equal to EB, and MP to FD; wherefore GK is to HO, as LN to MP. If, therefore, GK be greater than HO, LN is greater than MP; and if equal, equal; and if less, less (*A.* v.).

But let HO, MP be equimultiples of EB, FD:



And because (*Hyp.*) AE is to EB, as CF to FD, and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; if GK be greater than HO, LN is greater than MP; and if equal, equal; and if less, less (5 *Def.* v.); which was likewise shown in the preceding case.

If therefore GH be greater then KO, taking KH from both, GK is greater than HO; wherefore also LN is greater than MP; and consequently, adding NM to both, LM is greater than NP: therefore, if GH be greater than KO, LM is greater than NP.

In like manner it may be shown, that if GH be equal to KO, LM is equal to NP; and if less, less.

And in the case in which KO is not greater than KH, it has been shown that GH is always greater than KO, and likewise LM than NP: but GH, LM are any equimultiples of AB, CD, and KO, NP are any whatever of BE, DF; therefore (5 *Def.* v.), as AB is to BE, so is CD to DF. If then magnitudes, etc. Q. E. D.

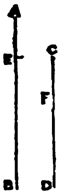
PROPOSITION XIX.

THEOR. *If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole to the whole.*

Let the whole AB, be to the whole CD, as AE, a magnitude taken from AB, to CF, a magnitude taken from CD; the remainder EB shall be to the remainder FD, as the whole AB to the whole CD.

Because AB is to CD, as AE to CF (*Hyp.*); likewise, alternately (16. v.), BA is to AE, as DC to CF:

And because, if magnitudes taken jointly be proportionals, they are also proportionals (17. v.) when taken separately; therefore, as BE is to EA, so is DF to FC; and alternately, as BE is to DF, so is EA to FC:

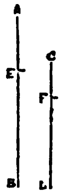


But, as AE to CF, so, by the hypothesis, is AB to CD; therefore also BE, the remainder, shall be to the remainder DF (11. v.), as the whole AB to the whole CD. Wherefore, if the whole, etc. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder likewise is to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

PROPOSITION E.

Theor. *If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second as the third to its excess above the fourth.*



Let AB be to BE, as CD to DF; then BA is to AE, as DC to CF.

Because AB is to BE, as CD to DF, by division (17. v.), AE is to EB, as CF to FD;

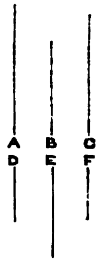
And by inversion (*B.* v.), BE is to EA, as DF to FC:

Wherefore, by composition (18. v.), BA is to AE, as DC is to CF. If, therefore, four, etc. Q. E. D.

PROPOSITION XX.

Theor. *If there be three magnitudes, and other three, which, taken two and two, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.*

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the same ratio, viz. as A is to B, so is D to E; and as B to C, so is E to F: if A be greater than C, D shall be greater than F; and if equal, equal; and if less, less.



Because A is greater than C, and B is any other magnitude, and that the greater has to the same magnitude a greater ratio than the less has to it (8. v.); therefore A has to B a greater ratio than C has to B:

But as D is to E, so is A to B; therefore (13. v.) D has to E a greater ratio than C to B:

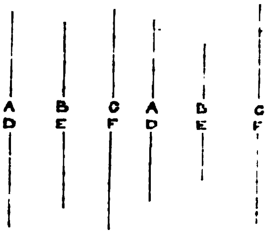
And because B is to C, as E to F, by inversion, C is to B, as F is to E (*B.* v.):

And D was shown to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E (*Cor.* 13. v.). But the magnitude which has a greater ratio than another to the same magnitude, is the greater of the two (10. v.); D is therefore greater than F.

Secondly, let A be equal to C; D shall be equal to F:

Because A and C are equal to one another, A is to B, as C is to B (7. v.):

But A is to B, as D to E; and C is to B, as F to E; wherefore D is to E, as F to E (11. v.); and therefore D is equal to F (9. v.):



Next, let A be less than C; D shall be less than F:

For C is greater than A; and, as was shown in the first case, C is to B, as F to E; and in like manner, B is to A, as E to D; therefore F is greater than D, by the first case; and therefore D is less than F. Therefore, if there be three, etc. Q. E. D.

PROPOSITION XXI.

Theor. *If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.*

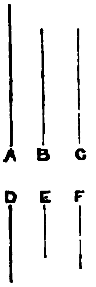
Let A, B, C be three magnitudes, and D, E, F other three, which have the same ratio, taken two and two, but in a cross order; viz. as A is to B, so is E to F, and as B is to C, so is D to E. If A be greater than C, D shall be greater than F; and if equal, equal; and if less, less.

Because A is greater than C, and B is any other magnitude, A has to B a greater ratio (8. v.) that C has to B:

But as E to F, so is A to B; therefore (13. v.) E has to F a greater ratio than C to B:

And because B is to C, as D to E, by inversion, C is to B, as E to D:

And E was shown to have to F a greater ratio than C to B; therefore E has to F a greater ratio than E to (*Cor.* 13. v.) D:



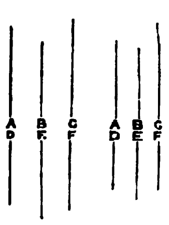
But the magnitude to which the same has a greater ratio than it has to another, is the less of the two (10. v.); F therefore is less than D; that is, D is greater than F.

Secondly, let A be equal to C; D shall be equal to F.

Because A and C are equal, A is (7. v.) to B, as C is to B:

But A is to B, as E to F; and C is to B, as E to D; wherefore E is to F, as E to D (11. v.); and therefore D is equal to F (9. v.).

Next, let A be less than C: D shall be less than F. For C is greater than A; and, as was shown, C is to B, as E to D, and in like manner B is to A, as F to E; therefore F is greater than D, by case first; and therefore, D is less than F. Therefore, if there be three, etc. Q. E. D.



PROPOSITION XXII.

THEOR. *If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first of the others has to the last.* N.B. This is usually cited by the words " ex æquali," or " ex æquo.".

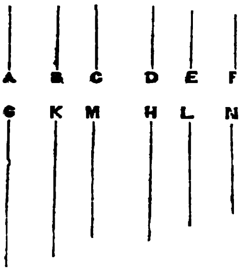
First, let there be three magnitudes, A B, C, and as many others, D, E, F, which, taken two and two, have the same ratio; that is, such that A is to B, as D to E; and as B is to C, so is E to F: A shall be to C, as D to F.

Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N:

Then, because A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; as G is to K, so is (4. v.) H to L:

For the same reason, K is to M, as L to N: and because there are three magnitudes G, K, M, and other three H, L, N, which, two and two, have the same ratio; if G be greater than M, H is greater than N; and if equal, equal; and if less, less (20. v.):

And G, H are any equimultiples whatever of A, D (*Constr.*), and M, N are any equimultiples of C, F; therefore (*Def.*), as A is to C, so is D to F.



Next, let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which, two and two, have the same ratio, viz., as A is to B, so is E to F; and as B to C, so F to G; and as C to D, so G to H: A shall be to D, as E to H.

A. B. C. D.
E. F. G. H.

Because A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio; by the foregoing case, A is to C, as E to G:

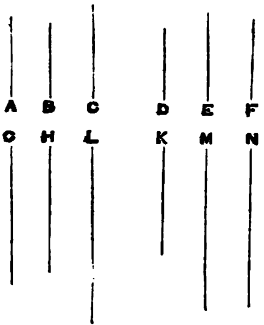
But C is to D, as G to H; wherefore again, by the first case, A is to D, as E to H; and so on, whatever be the number of magnitudes. Therefore, if there be any number, etc. Q. E. D.

PROPOSITION XXIII.

THEOR. *If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes, the same ratio which the first of the others has to the last.* N. B. This is usually cited by the words " ex æquali in proportione perturbatâ;" or " ex æquo perturbato."

First, let there be three magnitudes A, B, C, and other three, D, E, F, which, taken two and two in a cross order, have the same ratio; that is, such, that A is to B, as E to F; and as B is to C, so is D to E: A is to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, F any equimultiples whatever L, M, N:



And because G, H are equimultiples o A, B, and that magnitudes have the same ratio which their equimultiples have (15. v.); as A is to B, so is G to H:

And for the same reason, as E is to F, so is M to N:

But as A is to B, so is E to F; as therefore G is to H, so is M to N (11. v.).

And because as B is to C, so is D to E, and that H, K are equimultiples of B, D, and L, M of C, E; as H is to L, so is (4. v.) K to M.

And it has been shown that G is to H, as M to N;

Then, because there are three magnitudes G, H, L, and other three K, M, N, which have the same ratio taken two and two in a cross order; if G be greater than L, K is greater than N; and if equal, equal; and if less, less (21. v.):

And G, K are any equimultiples whatever of A, D; and L, N any whatever of C, F; as, therefore, A is to C (5 *Def.* v.), so is D to F.

Next, let there be four magnitudes A, B, C, D, and other four E, F, G, H, which, taken two and two in a cross order, have the same ratio, viz., A to B, as G to H; B to C, as F to G; and C to D, as E to F: A is to D, as E to H.

A. B. C. D.
E. F. G. H.

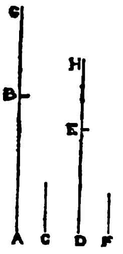
Because A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a cross order, have the same ratio; by the first case, A is to C, as F to H:

But C is to D, as E is to F; wherefore again, by the first case, A is to D, as E to H: and so on, whatever be the number of magnitudes. Therefore, if there be any number, etc. Q. E. D.

PROPOSITION XXIV.

THEOR. *If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth, the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.*

Let AB the first have to C the second, the same ratio which DE the third has to F the fourth; and let BG the fifth have to C the second, the same ratio which EH the sixth has to F the fourth: AG the first and fifth together, shall have to C the second, the same ratio which DH the third and sixth together, has to F the fourth.



Because BG is to C, as EH to F; by inversion, C is to BG (*B.* v.), as F to EH:

And because, as AB is to C, so is DE to F; and as C to BG, so F to EH; ex æquali (22. v.), AB is to BG, as DE to EH:

And because these magnitudes are proportionals, they shall likewise be proportionals when taken jointly (18. v.); as therefore AG is to GB, so is DH to HE: but as GB (*Hyp.*) to C, so is HE to F: therefore ex æquali (22. v.), as AG is to C, so is DH to F. Wherefore, if the first, etc. Q. E. D.

Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude, the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

PROPOSITION XXV.

Theor. *If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.*

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greatest of them, and consequently F the least (*A.* and 14. v.): AB together with F, are greater than CD together with E.

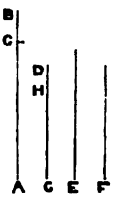
Take AG equal to E, and CH equal to F:

Then because as AB is to CD, so is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG to CH:

And because AB the whole, is to the whole CD, as AG is to CH, likewise the remainder GB shall be to the remainder HD, as the whole AB is to the whole (19. v.) CD:

But AB is greater than CD; therefore (*A.* v.) GB is greater than HD:

And because AG is equal to E, and CH to F; AG and F together, are equal to CH and E together:



If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD; AB and F together, are greater than CD and E. Therefore, if four magnitudes, etc. Q. E. D.

PROPOSITION F.

Theor. *Ratios which are compounded of the same ratios, are the same with one another.*

Let A be to B, as D to E; and B to C, as E to F: the ratio which is compounded of the ratios of A to B, and B to C, which, by the definition of compound ratio, is the ratio of A to C, is the same with the ratio of D to F, which, by the same definition, is compounded of the ratios of D to E, and E to F.

A. B. C.
D. E. F.

Because there are three magnitudes A, B, C, and three others D, E, F, which, taken two and two in order, have the same ratio, ex æquali A is to C, as D to F (22. v.).

Next, let A be to B, as E to F, and B to C, as D to E; therefore *ex æquali in proportione perturbatâ* (23. v.), A is to C, as D to F; that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the same with the ratio of D to F, which is compounded of the ratios of D to E, and E to F. And in like manner the proposition may be demonstrated, whatever be the number of ratios in either case.

A. B. C.
D. E. F.

EXERCISES ON BOOK V.

1. Certain of the following properties have been proved in the Fifth Book, respecting four proportional magnitudes A, B, C, D: let the student point out which and where; and prove the remaining ones.

Hypothesis, $A : B :: C : D$;
then (1) $A : C :: B : D$;
(2) $B : A :: D : C$;
(3) $B : D :: A : C$;
(4) $A \pm B : A :: C \pm D : C$;
(5) $A \pm B :: B :: C \pm D : D$;
(6) $A \pm B : C \pm D :: A : C$;
(7) $A \pm B : C \pm D :: B : D$;
(8) $A + B : A - B :: C + D : C - D$;
(9) $A + B : C + D :: A - B : C - D$.

2. Some of these are true when A, B are of a species different from that of C, D: let the student say which, and tell why.

3. Let m and n be any numbers, rational or irrational: then

$$mA : nB :: mC : nD.$$

4. If $mA = A'$, $nB = B'$, $mC = C'$, and $nD = D'$ show that the proportions (1) to (9) are true of A', B', C', D'.

5. If $A : B :: B : C$; then $A \pm B$ is greater or less than $B \pm C$, according as A is greater or less than B.

6. If there be several ratios, each of which is greater or less than another ratio, then the ratio of the sum of the antecedents of the first named ratios to the sum of the consequents will be respectively greater or less than that other ratio.

7. If the first of four magnitudes have a greater ratio to the second than the third has to the fourth; then,

(*a*) The second will have to the first a less ratio than the fourth to the third:—

(*b*) The first will have to the third a greater ratio than the second has to the fourth:—

(*c*) If the first be the greatest, the fourth will be the least, and *vice versâ*:—

(*d*) The first and second together will have to the second a greater ratio than the third and fourth together have to the fourth.

8. Prove the converses of these; and in all the cases show whether the truth be confined to the case of the four magnitudes being of the same kind or not.

9. If two magnitudes which have a ratio be each increased by equal magnitudes; then the ratio of the magnitudes thus composed will be

increased or diminished according as the original antecedent is less or greater than the consequent.

10. Show also what takes place if the original antecedent and consequent be diminished by equal magnitudes.

11. (*a.*) If any number of magnitudes be continued proportionals, their successive differences will also be proportionals.

(*b.*) Are they continued proportionals?

12. The ratio compounded of the ratios in the preceding proposition is greater than any of the component ratios.

13. (*a.*) Generally, a ratio compounded of ratios the whole of which are of greater or less inequality will be itself a ratio of respectively greater or less inequality; and likewise respectively greater or less than each of the component ratios.

(*b*) What is the ratio compounded of several ratios of equality?

14. If four magnitudes be proportionals, the ratio compounded of the ratio of one pair and the inverse ratio of the other pair is a ratio of equality.

15. If three magnitudes be proportionals, the two extreme terms together are greater than double the mean term.

[This is sometimes expressed by saying that the arithmetical mean is greater than the geometrical.]

16. Let any number of pairs of magnitudes have the same ratio:— Also, let any number of the antecedents be combined in any possible way by addition or subtraction, or both; and the corresponding consequents be combined in the same order:—

And likewise, let any other combination of other antecedents, or of the same, or of some of the former and some of the other, be similarly made, and the same combination of the corresponding consequents;—

Then the first combined antecedents will have to the corresponding combined consequents the same ratio that the latter has to the latter.

17. Of what proposition in the Fifth Book is the preceding an extension? and what conditions are essential amongst the magnitudes so combined, as regards *species*?

18. Let A : B : C : D : . . . . . : N be continued porportionals; then also will

$$A \pm B : B \pm C : C \pm D : \ . \ . \ . \ . \ . : M \pm N$$

be continued proportionals.

19. Make the same hypothesis as in the preceding proposition; let also a series of magnitudes be taken increasing or decreasing by the difference of A and B, according as A is less or greater than B, of which A is the first: if these be A, B, C′, D′, ... N′, then throughout the series C′, D′, . . . N′ will respectively be greater than C, D, . . . N.

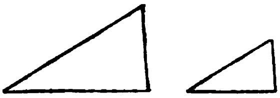
20. If two multiplicate ratios of the same number of ratios (as both duplicate, both triplicate, etc.), be the same, then the fundamental ratios will be also the same.

[This theorem is important, and *must be* demonstrated by the pupil. The method *ex absurdo* is perhaps the most simple.]

BOOK VI.

DEFINITIONS.

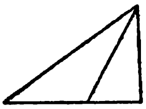
1. **Similar** rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



2. "Reciprocal figures, viz. triangles and parallelograms, are such "as have their sides about two of their angles proportionals in such a " manner, that a side of the first figure is to a side of the other, as the "remaining side of this other is to the remaining side of the first."

3. A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment as the greater segment is to the less.

4. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.

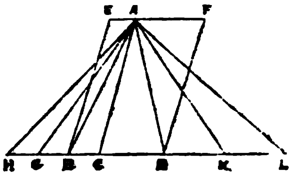


PROPOSITIONS.

PROPOSITION I.

Theor. *Triangles and parallelograms of the same altitude are one to another as their bases.*

Let the triangles ABC, ACD, and the parallelograms EC, CF, have the same altitude, viz. the perpendicular drawn from the point A to BD; then, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.



Produce BD both ways to the points H, L, and (3. I.) take any number of straight lines BG, GH, each equal to the base BC; and DK, KL, any number of them, each equal to the base CD; and join AG, AH, AK, AL:

Then, because CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all equal (38. I.): therefore, whatever multiple the base HC is of the base BC, the same multiple is the triangle AHC of the triangle ABC.

For the same reason, whatever multiple the base LC is of the base CD, the same multiple is the triangle ALC of the triangle ADC:

And if the base HC be equal to the base CL, the triangle AHC is also equal to the triangle ALC (38. I.): and if the base HC be greater than the base CL, likewise the triangle AHC is greater than the triangle ALC; and if less, less:

Therefore, since there are four magnitudes, viz. the two bases BC, CD, and the two triangles ABC, ACD; and of the base BC, and the triangle ABC, the first and third, any equimultiples whatever have been taken, viz. the base HC and triangle AHC; and of the base CD and triangle ACD, the second and fourth, have been taken any equimultiples whatever, viz. the base CL and triangle ALC; and that it has been shown that, if the base HC be greater than the base CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if less, less: therefore (5 *Def.* V.), as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

And because the parallelogram CE is double of the triangle ABC, (41. I.) and the parallelogram CF double of the triangle ACD, and that magnitudes have the same ratio which their equimultiples have (15. V.); as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF:

And because it has been shown, that, as the base BC is to the base CD, so is the triangle ABC to the triangle ACD; and as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF; therefore, as the base BC is to the base CD, so is (11. V.) the parallelogram EC to the parallelogram CF. Wherefore, triangles, etc. Q. E. D.

COR. From this it is plain, that triangles and parallelograms that have equal altitudes, are one to another as their bases.

Let the figures be placed so as to have their bases in the same straight line; and having drawn perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are (33. I.), because the perpendiculars are both equal and parallel (28. I.) to one another. Then, if the same construction be made as in the proposition, the demonstration will be the same.

PROPOSITION II.

THEOR. *If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally: and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.*

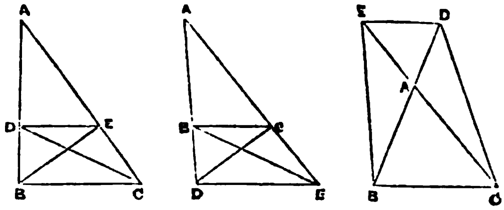
Let DE be drawn parallel to BC, one of the sides of the triangle ABC: BD is to DA, as CE to EA.

Join BE, CD:

Then the triangle BDE is equal to the triangle CDE (37. I.), because they are on the same base DE, and between the same parallels DE, BC.

ADE is another triangle; and equal magnitudes have the same ratio to the same magnitude (7. v.); therefore, as the triangle BDE is to the triangle ADE, so is the triangle CDE to the triangle ADE:

But as the triangle BDE to the triangle ADE, so is (1. vi.) BD to DA, because, having the same altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as the triangle CDE to the triangle ADE, so is CE to EA:



Therefore, as BD to DA, so is CE to EA (11. v.).

Next, let the sides AB, AC of the triangle ABC, or these produced be cut proportionally in the points D, E, that is, so that BD be to DA, as CE to EA; and join DE: DE is parallel to BC.

The same construction being made, because as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE (1. vi.); and as CE to EA, so is the triangle CDE to the triangle ADE; therefore (11. v.) the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the same ratio to the triangle ADE; and therefore (9. v.) the triangle BDE is equal to the triangle CDE; and they are on the same base DE: but equal triangles on the same base are between the same parallels (39. i.); therefore DE is parallel to BC. Wherefore, if a straight line, etc. Q. E. D.

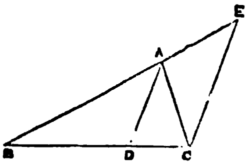
PROPOSITION III.

Theor. *If the angle of a triangle be divided into two equal angles, by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another: and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section divides the vertical angle into two equal angles.*

Let the angle BAC of any triangle ABC be divided into two equal triangles by the straight line AD; BD is to DC, as BA to AC.

Through the point C draw CE parallel (31. i.) to DA, and let BA produced meet CE in E.

Because the straight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD (29. i.): but CAD, by the hypothesis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE.



Again, because the straight line BAE meets the parallels AD, EC, the outward angle BAD is equal (29. i.) to the inward and opposite angle AEC: but the angle ACE has been proved equal to the angle BAD; therefore also ACE is equal to the angle AEC, and consequently the side AE is equal to the side (6. I.) AC:

And because AD is drawn parallel to one of the sides of the triangle BCE, viz. to EC; BD is to DC, as BA to AE (2. VI.):

But AE is equal to AC; therefore, as BD to DC, so is BA to AC (7. V.).

Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the straight line AD.

The same construction being made; because, as BD to DC, so is BA to AC; and as BD to DC, so is BA to AE, because AD is parallel to EC (2. VI.); therefore BA is to AC, as BA to AE (11. V.):

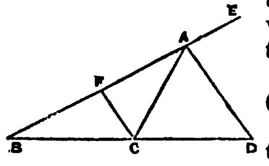
Consequently AC is equal to AE (9. V.), and the angle AEC is therefore equal to the angle ACE (5. I.):

But the angle AEC is equal to the outward and opposite angle BAD; and the angle ACE is equal to the alternate angle CAD (29. I.): wherefore also the angle BAD is equal to the angle CAD; therefore the angle BAC is cut into two equal angles by the straight line AD. Therefore, if the angle, etc. Q. E. D.

PROPOSITION A.

THEOR. *If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to one another; and if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.*

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the straight line AD which meets the base produced in D: BD is to DC, as BA to AC.



Through C, draw CF parallel to AD (31. I.):

And because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (29. I.): but CAD is equal to the angle DAE (*Hyp.*); therefore also DAE is equal to the angle ACF.

Again, because the straight line FAE meets the parallels AD, FC, the outward angle DAE is (29. I.) equal to the inward and opposite angle CFA: but the angle ACF has been proved equal to the angle DAE; therefore also the angle ACF is equal to the angle CFA, and consequently the side AF is equal to the side AC (6. I.):

And because AD is parallel to FC, a side of the triangle BCF; BD is to DC, as BA to AF (2. VI.):

But AF is equal to AC; as therefore BD is to DC (7. VI.), so is BA to AC.

Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

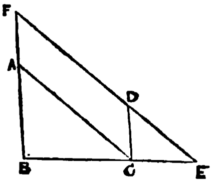
The same construction being made; because BD is to DC, as BA to AC: and that BD is also to DC, as BA to AF (2. VI.); therefore BA is to AC, as BA to AF (11. V.); wherefore AC is equal to AF (9. V.), and the angle AFC equal (5. I.) to the angle ACF:

But the angle AFC is equal to the outward angle EAD (29. I.), and the angle ACF to the alternate angle CAD; therefore also EAD is equal to the angle CAD. Wherefore, if the outward, etc. Q. E. D.

PROPOSITION IV.

THEOR. *The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.*

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC; and consequently (32. I. *and* 3 *Ax.*) the angle BAC equal to the angle CDE: the sides about the equal angles of the triangles ABC, DCE, are proportionals; and those are the homologous sides, which are opposite to the equal angles.



Let the triangle DCE be placed so that its side CE may be contiguous to BC, and in the same straight line with it: and because the angles ABC, ACB are together less than two right angles (17. I.), ABC, and DEC, which is equal (*Hyp.*) to ACB, are also less than two right angles; wherefore BA, ED produced shall meet (12. *Ax.* I.): let them be produced and meet in the point F:

And because the angle ABC is equal to the angle DCE, BF is parallel (28. I.) to CD: again, because the angle ACB is equal to the angle DEC, AC is parallel to FE (28. I.); therefore FACD is a parallelogram, and consequently AF is equal to CD, and AC to FD (34. I.):

And because AC is parallel to FE, one of the sides of the triangle FBE, BA is to AF, as BC to CE (2. VI.): but AF is equal to CD; therefore (7. V.), as BA to CD, so is BC to CE; and alternately (16. V.), as AB to BC, so is DC to CE:

Again, because CD is parallel to BF, as BC to CE, so is FD to DE (2. VI.): but FD is equal to AC; therefore (7. V.), as BC to CE, so is AC to DE: and alternately (16. V.), as BC to CA, so CE to ED:

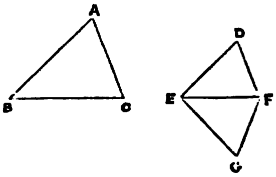
Therefore, because it has been proved that AB is to BC, as DC to CE; and as BC to CA, so CE to ED: ex æquali (22. V.), BA is to AC, as CD to DE. Therefore the sides, etc. Q. E. D.

PROPOSITION V.

THEOR. *If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.*

Let the triangles ABC, DEF have their sides proportionals, so that

AB is to BC, as DE to EF; and BC to CA, as EF to FD; and consequently, ex æquali, BA to AC, as ED to DF: the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides, viz., the angle ABC equal to the angle DEF, and BCA to EFD, and also BAC to EDF.



At the points E, F, in the straight line EF, make (23. I.) the angle FEG equal to the angle ABC, and the angle EFG equal to BCA;

Wherefore the remaining angle BAC is equal to the remaining angle EGF (32. I.); and the triangle ABC is therefore equiangular to the triangle GEF; and consequently they have their sides opposite to the equal angles proportionals (4. VI.). Wherefore, as AB to BC, so is GE to EF: but as AB to BC (*Hyp.*), so is DE to EF; therefore, as DE to EF, so (11. V.) GE to EF; therefore DE and GE have the same ratio to EF, and consequently are equal (9. V.):

For the same reason, DF is equal to FG:

And because, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two sides DE, EF are equal to the two GE, EF, each to each; and the base DF is equal to the base GF; therefore the angle DEF is equal (8. I.) to the angle GEF, and the other angles to the other angles which are subtended by the equal sides (4. I.): wherefore the angle DFE is equal to the angle GFE, and EDF to EGF:

And because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF: for the same reason, the angle ACB is equal to the angle DFE, and the angle at A equal to the angle at D: therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the sides, etc. Q. E. D.

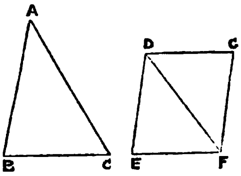
PROPOSITION VI.

THEOR. *If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.*

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the sides about those angles proportionals; that is, BA to AC, as ED to DF: the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the straight line DF, make (23. I.) the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB:

Wherefore the remaining angle at B is equal to the remaining one at G (32. I.); and consequently the triangle ABC is equiangular to the triangle DGF; and therefore as BA to AC, so is (4. VI.) GD to DF: but by the hypothesis, as BA to AC, so is ED to DF; as therefore ED to DF, so is (11. V.) GD to DF; wherefore ED is equal (9. V.) to DG:



And DF is common to the two triangles EDF, GDF; therefore the two sides ED, DF are equal to the two sides GD, DF, each to each; and the angle EDF (*Constr.*) is equal to the angle GDF; wherefore the base EF is equal to the base FG (4. I.), and the triangle EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides; therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E:

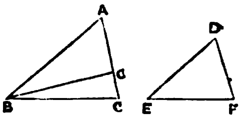
But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE: and the angle BAC is equal to the angle EDF (*Hyp.*); wherefore also the remaining angle at B is equal to the remaining angle at E: therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, etc. Q. E. D.

PROPOSITION VII.

THEOR. *If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.*

Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz., the angle BAC to the angle EDF, and the sides about two other angles ABC, DEF proportionals, so that AB is to BC, as DE to EF; and, in the first case, let each of the remaining angles at C, F be less than a right angle: the triangle ABC is equiangular to the triangle DEF, viz., the angle ABC is equal to the angle DEF, and the remaining angle at C equal to the remaining angle at F.

For if the angles ABC, DEF be not equal, one of them is greater than the other: let ABC be the greater, and at the point B, in the straight line AB, make the angle ABG equal to the angle (23. I.) DEF:



And because the angle at A is equal to the angle at D, and the angle ABG to the angle DEF, the remaining angle AGB is equal (32. I.) to the remaining angle DFE: therefore the triangle ABG is equiangular to the triangle DEF; wherefore (4. VI.) as AB is to BG, so is DE to EF: but as DE to EF, so, by hypothesis, is AB to BC; therefore as AB to BC, so is AB to BG (11. V.): and because AB has the same ratio to each of the lines BC, BG; BC is equal (9. V.) to BG; and therefore the angle BGC is equal to the angle BCG (5. I.):

But the angle BCG is, by hypothesis, less than a right angle; therefore also the angle BGC is less than a right angle, and the adjacent angle AGB must be greater than a right angle (13. I.):

But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle: but, by the hypothesis, it is less than a right angle; which is absurd. Therefore the angles ABC, DEF are not unequal, that is, they are equal: and the angle at A is equal (*Hyp.*) to the angle at D; wherefore the re-

PROPOSITION XI.

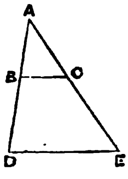
Prob. *To find a third proportional to two given straight lines.*

Let AB, AC be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to AB, AC.

Produce AB, AC, to the points D, E; and make BD equal to AC (3. I.); and having joined BC, through D, draw DE parallel to it (31. I.):

Because BC is parallel to DE, a side of the triangle ADE, AB is (2. VI.) to BD, as AC to CE:

But BD is equal to AC; as therefore AB to (7. V.) AC, so is AC to CE. Wherefore, to the two given straight lines AB, AC, a third proportional CE is found. Which was to be done.



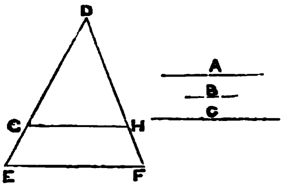
PROPOSITION XII.

Prob. *To find a fourth proportional to three given straight lines.*

Let A, B, C be three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines, DE, DF, containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel (31. I.) to it through the point E:

And because GH is parallel to EF, one of the sides of the triangle DEF, DG is to GE, as DH to HF (2. VI.): but DG is equal to A, GE to B, and DH to C; therefore, as A is to B (7. V.), so is C to HF. Wherefore, to the three given straight lines A, B, C, a fourth proportional HF is found. Which was to be done.



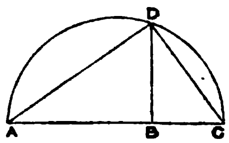
PROPOSITION XIII.

Prob. *To find a mean proportional between two given straight lines.*

Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line, and upon AC describe the semicircle ADC; and from the point B draw (11. I.) BD at right angles to AC, and join AD, DC.

Because the angle ADC in a semicircle is a right angle (31. III.), and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB, BC the segments of the base (*Cor.* 8. VI.). Therefore, between the two given straight lines AB, BC, a mean proportional DB is found. Which was to be done.



PROPOSITION XIV.

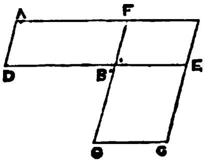
THEOR. *Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

Let AB, BC be equal parallelograms which have the angles at B equal, and let the sides DB, BE be placed in the same straight line; wherefore also FB, BG are in one straight line (14. I.). The sides of the parallelograms AB, BC about the equal angles are reciprocally proportional; that is, DB is to BE, as GB to BF.

Complete the parallelogram FE:

And because the parallelogram AB is equal to BC (*Hyp.*), and that FE is another parallelogram, AB is to FE, as BC to FE (7. V.):

But as AB to FE, so is the base DB to BE (1. VI.); and as BC to FE, so is the base of GB to BF; therefore, as DB to BE, so is GB to BF (11. V.). Wherefore, the sides of the parallelograms AB, BC about their equal angles are reciprocally proportional.



But, let the sides about the equal angles be reciprocally proportional, viz. as DB to BE, so GB to BF; the parallelogram AB is equal to the parallelogram BC.

Because, as DB to BE, so is GB to BF; and as DB to BE (1. VI.), so is the parallelogram AB to the parallelogram FE; and as GB to BF, so is the parallelogram BC to the parallelogram FE; therefore, (11. V.) as AB to FE, so BC to FE: wherefore the parallelogram AB is equal (9. V.) to the parallelogram BC. Therefore, equal parallelograms, etc. Q. E. D.

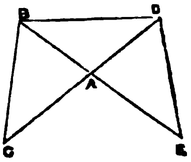
PROPOSITION XV.

THEOR. *Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the sides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed so that their sides CA, AD be in one straight line; wherefore also EA and AB are in one straight line (14. I.); and join BD.

Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, so is triangle EAD to triangle DAB (7. V.):



But as triangle CAB to triangle BAD, so is the base CA to AD (1. VI.); and as triangle EAD to triangle DAB, so is the base EA to AB (1. VI.); as therefore CA to AD, so is EA to AB (11. V.):

wherefore the sides of the triangles ABC, ADE about the equal angles are reciprocally proportional.

But let the sides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

Having joined BD as before; because, as CA to AD, so is EA to AB; and as CA to AD, so is triangle ABC to triangle BAD (1. vi.); and as EA to AB, so is triangle EAD to triangle BAD (1. vi.); therefore (11. v.) as triangle BAC to triangle BAD, so is triangle EAD to triangle BAD; that is, the triangles BAC, EAD have the same ratio to the triangle BAD: wherefore the triangle ABC is equal (9. v.) to the triangle ADE. Therefore, equal triangles, etc. Q. E. D.

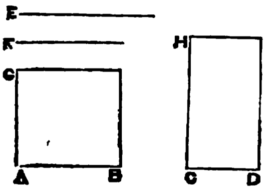
PROPOSITION XVI.

Theor. *If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.*

Let the four straight lines AB, CD, E, F, be proportionals, viz. as AB to CD, so E to F; the rectangle contained by AB, F, is equal to the rectangle contained by CD, E.

From the points A, C, draw (11. i.) AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E; and complete the parallelograms BG, DH:

Because, as AB to CD, so is E to F; and that E is equal to CH, and F to AG; AB is (7. v.) to CD, as CH to AG: therefore the sides of the parallelograms BG, DH about the equal angles, are reciprocally proportional but parallelograms which have their sides about equal angles reciprocally proportional are equal to one another (14. vi.); therefore the parallelogram BG is equal to the parallelogram DH:



And the parallelogram BG is contained by the straight lines AB, F, because AG is equal to F; and the parallelogram DH is contained by CD and E, because CH is equal to E; therefore the rectangle contained by the straight lines AB, F is equal to that which is contained by CD and E.

And if the rectangle contained by the straight lines AB, F, be equal to that which is contained by CD, E; these four lines are proportionals, viz. AB is to CD, as E to F.

The same construction being made; because the rectangle contained by the straight lines AB, F, is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, because AG is equal to F; and the rectangle DH by CD, E, because CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH;

And they are equiangular: but the sides about the equal angles of equal parallelograms are reciprocally proportional (14. vi.): wherefore, as AB to CD, so is CH to AG: and CH is equal to E, and AG to F; as therefore AB is to CD (7. v.), so is E to F. Wherefore, if four, etc. Q. E. D.

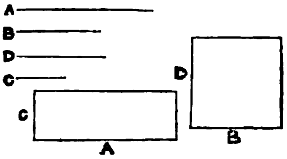
PROPOSITION XVII.

Theor. *If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean; and if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.*

Let the three straight lines A, B, C be proportionals, viz. as A to B, so B to C; the rectangle contained by A, C is equal to the square of B:

Take D equal to B;

And because as A to B, so B to C, and that B is equal to D; A is (7. v.) to B, as D to C: but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means (16. vi.); therefore the rectangle contained by A, C, is equal to that contained by B, D: but the rectangle contained by B, D is the square of B; because B is equal to D; therefore the rectangle contained by A, C is equal to the square of B.



And if the rectangle contained by A, C be equal to the square of B; A is to B, as B to C.

The same construction being made; because the rectangle contained by A, C is equal to the square of B, and the square of B is equal to the rectangle contained by B, D, because B is equal to D; therefore the rectangle contained by A, C is equal to that contained by B, D: but if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals (16. vi.); therefore A is to B, as D to C: but B is equal to D; wherefore, as A to B, so B to C. Therefore, if three straight lines, etc. Q. E. D.

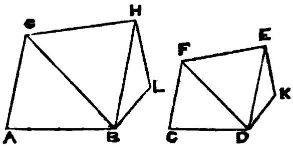
PROPOSITION XVIII.

Prob. *Upon a given straight line to describe a rectilineal figure, similar, and similarly situated, to a given rectilineal figure.*

Let AB be the given straight line, and CDEF the given rectilineal figure of four sides; it is required, upon the given straight line AB, to describe a rectilineal figure, similar, and similarly situated, to CDEF.

Join DF, and at the points A, B, in the straight line AB, make (23. i.) the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF;

Therefore the remaining angle CFD is equal to the remaining angle AGB (32. i.): wherefore the triangle FCD is equiangular to the triangle GAB: again, at the points G, B, in the straight line GB, make (23. i.) the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH:



Then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE: for the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED; therefore the rectilineal figure ABHG is equiangular to CDEF:

But likewise these figures have their sides about the equal angles proportionals; because the triangles GAB, FCD being equiangular, BA is (4. VI.) to AG, as DC to CF; and because AG is to GB, as CF to FD; and as GB to GH, so, by reason of the equiangular triangles BGH, DFE, is FD to FE; therefore, ex æquali (22. V.), AG is to GH, as CF to FE:

In the same manner it may be proved that AB is to BH, as CD to DE: and GH is to HB, as FE to ED (4. VI.).

Wherefore, because the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another (1 *Def.* VI.).

Next, let it be required to describe upon a given straight line AB, a rectilineal figure, similar, and similarly situated, to the rectilineal figure CDKEF.

Join DE, and upon the given straight line AB describe the rectilineal figure ABHG, similar, and similarly situated, to the quadrilateral figure CDEF, by the former case: and at the points B, H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore, the remaining angle at K is equal to the remaining angle at L (32. I.):

And because the figures ABHG, CDEF are similar (*Constr.*), the angle GHB is equal to the angle FED: and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK: for the same reason, the angle ABL is equal to the angle CDK; therefore the five-sided figures AGHLB, CFEKD are equiangular:

And because the figures AGHB, CFED are similar, GH is to HB, as FE to ED; and as HB to HL, so is ED to EK (4. VI.); therefore, ex æquali (22. V.) GH is to HL as FE to EK:

For the same reason, AB is to BL as CD to DK:

And BL is to LH as (4. VI.) DK to KE, because the triangles BLH, DKE are equiangular:

Therefore, because the five-sided figures AGHLB, CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another: and in the same manner a rectilineal figure of six or more sides may be described upon a given straight line similar to one given, and so on. Which was to be done.

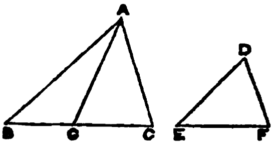
PROPOSITION XIX.

THEOR. *Similar triangles are to one another in the duplicate ratio of their homologous sides.*

Let ABC, DEF be similar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, so that the side BC is homologous to EF (12 *Def.* V.): the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF (11. VI.), so that BC is to EF, as EF to BG, and join GA:

Then, because, as AB to BC, so DE to EF; alternately (16. v.), AB is to DE, as BC to EF: but as BC to EF, so is EF to BG; therefore (11. v.), as AB to DE, so is EF to BG. Wherefore the sides of the triangles ABG, DEF, which are about the equal angles, are reciprocally proportional:



But triangles, which have the sides about two equal angles reciprocally proportional, are equal to one another (15. vi.); therefore the triangle ABG is equal to the triangle DEF:

And because as BC is to EF, so EF to BG; and that if three straight lines be proportionals, the first is said (10 *Def.* v.) to have to the third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF:

But as BC to BG, so is (1. vi.) the triangle ABC to the triangle ABG; therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: but the triangle ABG is equal to the triangle DEF; wherefore also the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore, similar triangles, etc. Q. E. D.

COR. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first, to a similar and similarly described triangle upon the second.

PROPOSITION XX.

THEOR. *Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.*

Let ABCDE, FGHKL be similar polygons, and let AB be the homologous side to FG: the polygons ABCDE, FGHKL may be divided into the same number of similar triangles, whereof each to each has the same ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL, the duplicate ratio of that which the side AB has to the side FG.

Join BE, EC, GL, LH:

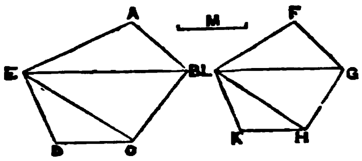
And because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL (1 *Def.* vi.), and BA is to AE, as GF to FL (1 *Def.* vi.):

Wherefore, because the triangles ABE, FGL have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle ABE is equiangular (6. vi.), and therefore similar, to the triangle FGL (4. vi.); wherefore the angle ABE is equal to the angle FGL:

And because the polygons are similar, the whole angle ABC is equal (1 *Def.* vi.) to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH:

And because the triangles ABE, FGL are similar, EB is to BA, as

LG to GF (1 *Def.* VI.); and also, because the polygons are similar, AB is to BC, as FG to GH (1 *Def.* VI.); therefore, ex æquali (22. V.), EB is to BC, as LG to GH; that is, the sides about the equal angles EBC, LGH are proportionals; therefore (6. VI.) the triangle EBC is equiangular to the triangle LGH, and similar to it (4. VI.); for the same reason, the triangle ECD likewise is similar to the triangle LHK: therefore the similar polygons ABCDE, FGHKL are divided into the same number of similar triangles.



Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the consequents FGL, LGH, LHK: and the polygon ABCDE has to the polygon FGHKL, the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL, the duplicate ratio (19. VI.) of that which the side BE has to the side GL:

For the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL; therefore, as the triangle ABE is to the triangle FGL, so (11. V.) is the triangle BEC to the triangle GLH.

Again, because the triangle EBC is similar (6. VI.) to the triangle LGH, EBC has to LGH, the duplicate ratio of that which the side EC has to the side LH (4. VI.):

For the same reason, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH: as therefore the triangle EBC to the triangle LGH, so is (11. V.) the triangle ECD to the triangle LHK:

But it has been proved, that the triangle EBC is likewise to the triangle LGH, as the triangle ABE to the triangle FGL;

Therefore, as the triangle ABE is to the triangle FGL, so is triangle EBC to triangle LGH, and triangle ECD to triangle LHK:

And therefore, as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents (12. V.); wherefore, as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL: but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the side AB has to the homologous side FG; therefore also the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG. Wherefore, similar polygons, etc. Q. E. D.

COR. 1. In like manner it may be proved, that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and it has already been proved in triangles; therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

COR. 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, AB has (10 *Def.* V.) to M the duplicate ratio of that which AB has to FG: but the four-sided figure or polygon upon AB, has to the four-sided figure or polygon upon FG, likewise the duplicate ratio of that which AB has to FG; therefore, as AB is to M, so is the figure upon AB to the figure upon FG; which was also proved in triangles (*Cor.* 19. VI.); therefore, universally, it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second

PROPOSITION XXI.

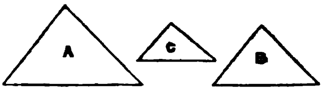
THEOR. *Rectilineal figures, which are similar to the same rectilineal figure, are also similar to one another.*

Let each of the rectilineal figures A, B be similar to the rectilineal figure C: the figure A is similar to the figure B.

Because A is similar to C, they are equiangular, and also have their sides about the equal angles proportionals (1 *Def.* VI.):

Again, because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals (1 *Def.* VI.):

Therefore the figures A, B are each of them equiangular to C, and have the sides about the equal angles of each of them, and of C, proportionals. Wherefore the rectilineal figures A and B are equiangular (1 *Ax.* I.), and have their sides about the equal angles proportionals (11. V.): therefore A is similar (1 *Def.* VI.) to B. Therefore, rectilineal figures, etc. Q. E. D.



PROPOSITION XXII.

THEOR. *If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals: and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.*

Let the four straight lines AB, CD, EF, GH be proportionals, viz. AB to CD as EF to GH: and upon AB, CD let the similar rectilineal figures KAB, LCD be similarly described; and upon EF, GH the similar rectilineal figures MF, NH, in like manner: the rectilineal figure KAB is to LCD, as MF to NH.

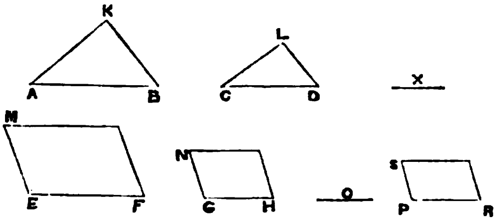
To AB, CD, take a third proportional (11. VI.) X; and to EF, GH, a third proportional O:

And because AB is to CD as EF to GH; and that CD is (11. V.) to X as GH to O; wherefore, ex æquali (22. V.), as AB to X, so EF to O:

But as AB to X, so is the rectilineal KAB to the rectilineal LCD, and as EF to O, so is (2 *Cor.* 20. VI.) the rectilineal MF to the rectilineal figure NH; therefore, as KAB to LCD, so (11. V.) is MF to NH.

And if the rectilineal KAB be to LCD as MF to NH; the straight line AB is to CD as EF to GH.

Make (12. VI.) as AB to CD, so EF to PR; and upon PR describe



(18. VI.) the rectilineal figure SR, similar and similarly situated to either of the figures MF, NH:

Then, because as AB to CD, so is EF to PR, and that upon AB, CD are described the similar and similarly situated rectilineals KAB, LCD, and upon EF, PR, in like manner, the similar rectilineals MF, SR; KAB is to LCD, as MF to SR: but, by the hypothesis, KAB is to LCD, as MF to NH; and therefore, the rectilineal MF having the same ratio to each of the two NH, SR, these are equal (9. V.) to one another: they are also similar, and similarly situated; therefore GH is equal to PR:

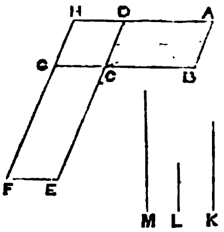
And because as AB to CD, so is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four straight lines, etc. Q. E. D.

PROPOSITION XXIII.

THEOR. *Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.*

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: the ratio of the parallelogram AC to the parallelogram CF, is the same with the ratio which is compounded of the ratios of their sides.

Let BC, CG be placed in a straight line; therefore DC and CE are also in a straight line (14. I.); and complete the parallelogram DG; and taking any straight line K, make (12. VI.) as BC to CG, so K to L; and as DC to CE, so make (12. VI.) L to M: therefore the ratios of K to L, and L to M, are the same with the ratios of the sides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded (*A. Def.* V.) of the ratios of K to L, and L to M: wherefore also K has to M the ratio compounded of the ratios of the sides:



And because as BC to CG, so is the parallelogram AC to the parallelogram CH (1. VI.): but as BC to CG, so is K to L; therefore K is (11. V.) to L, as the parallelogram AC to the parallelogram CH:

Again, because as DC to CE, so is the parallelogram CH to the parallelogram CF; but as DC to CE, so is L to M; wherefore L is (11. V.) to M, as the parallelogram CH to the parallelogram CF; therefore since it has been proved, that as K to L, so is the parallelogram AC to the parallelogram CH; and as L to M, so the parallelogram CH to the parallelogram CF; ex æquali (22. v.), K is to M, as the parallelogram AC to the parallelogram CF: but K has to M the ratio which is compounded of the ratios of the sides; therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides. Wherefore, equiangular parallelograms, etc. Q. E. D.

PROPOSITION XXIV.

THEOR. *The parallelograms about the diameter of any parallelogarm, are similar to the whole, and to one another.*

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: the parallelograms EG, HK are similar both to the whole parallelogram ABCD, and to one another.

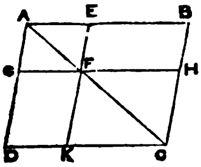
Because DC, GF are parallels, the angle ADC is equal (29. I.) to the angle AGF:

For the same reason, because BC, EF are parallels, the angle ABC is equal to the angle AEF:

And each of the angles BCD, EFG is equal to the opposite angle DAB (34 I.), and therefore are equal to one another; wherefore the parallelograms ABCD, AEFG are equiangular:

And because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to each other; therefore (4. VI.) as AB to BC, so is AE to EF:

And because the opposite sides of parallelograms are equal to one another (34. I.), AB (7. v.) is to AD, as AE to AG; and DC to CB, as GF to FE; and also CD to DA, as FG to GA; therefore the sides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore similar to one another (1 *Def.* VI.): for the same reason the parallelogram ABCD is similar to the parallelogram FHCK: wherefore each of the parallelograms GE, KH is similar to DB: but rectilineal figures which are similar to the same rectilineal figure, are also similar to one another (21. VI.); therefore the parallelogram GE is similar to KH. Wherefore the parallelograms, etc. Q. E. D.



PROPOSITION XXV.

PROB. *To describe a rectilineal figure which shall be similar to one, and equal to another, given rectilineal figure.*

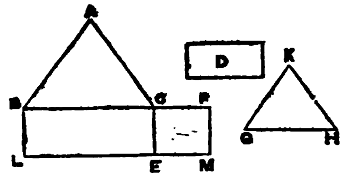
Let ABC be the given rectilineal figure to which the figure to be described is required to be similar, and D that to which it must be equal: it is required to describe a rectilineal figure similar to ABC, and equal to D.

Upon the straight line BC, describe (*Cor.* 45. I.) the parallelogram BE equal to the figure ABC; also upon CE describe (*Cor.* 45. I.) the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL; therefore BC and CF are in a straight line (29. I., 14. I.), as also LE and EM: between BC and CF find (13. VI.) a mean proportional GH; and upon GH describe (18. VI.) the rectilineal figure KGH, similar and similarly situated to the figure ABC:

And because BC is to GH as GH to CF; and if three straight lines be proportionals, as the first is to the third, so is (2 *Cor.* 20. VI.) the figure upon the first to the similar and similarly described figure upon the second; therefore as BC to CF, so is the rectilineal figure ABC to KGH:

But as BC to CF, so is (1. VI.) the parallelogram BE to the parallelogram EF; therefore, as the rectilineal figure ABC is to KGH, so is the parallelogram BE to the parallelogram EF (11. V.);

And the rectilineal figure ABC is equal (*Constr.*) to the parallelogram BE; therefore the rectilineal figure KGH is equal (14. V.) to the parallelogram EF: but EF is equal to the figure D; wherefore also KGH is equal to D; and it is similar to ABC. Therefore the rectilineal figure KGH has been described similar to the figure ABC, and equal to D. Which was to be done.

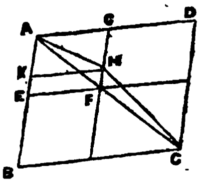


PROPOSITION XXVI.

THEOR. *If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.*

Let the parallelograms ABCD, AEFG be similar and similarly situated, and have the angle DAB common: ABCD and AEFG are about the same diameter.

For, if not, let if possible the parallelogram BD have its diameter AHC in a different straight line from AF, the diameter of the parallelogram EG; and let GF meet AHC in H; and through H draw HK parallel to to AD or BC;



Therefore the parallelograms ABCD, AKHG being about the same diameter, they are similar to one another (24. VI.): wherefore as DA to AB, so is (1 *Def.* VI.) GA to AK:

But because ABCD and AEFG are similar (*Hyp.*) parallelograms, as DA is to AB, so is GA to AE;

Therefore (11. V.) as GA to AE, so GA to AK: wherefore GA has the same ratio to each of the straight lines AE, AK; and, consequently, AK is equal (9. V.) to AE, the less to the greater, which is impossible; therefore ABCD and AKHG are not about the same diameter: wherefore ABCD and AEFG must be about the same diameter. Therefore, if two similar, etc. Q. E. D.

[As Propositions 27, 28, 29, of this Book are never referred to subsequently, they are not retained in the Course.]

PROPOSITION XXX.

Prob. *To cut a given straight line in extreme and mean ratio.*

Let AB be the given straight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point D, so that the rectangle contained by AB, BD, be equal to the square of AD (11. ii.):

Then, because the rectangle AB, BD is equal to the square of AD; as BA to AD, so is AD to DB (17. vi.): therefore AB is cut in extreme and mean ratio in D (3 *Def.* vi.). Which was to be done.

PROPOSITION XXXI.

Theor. *In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.*

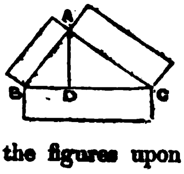
Let ABC be a right-angled triangle, having the right angle BAC: the rectilineal figure described upon BC is equal to the similar and similarly described figures upon BA, AC.

Draw the perpendicular AD:

Therefore, because in the right-angled triangle ABC, AD is drawn from the right angle at A, perpendicular to the base BC, the triangles ABD, ADC are similar to the whole triangle ABC, and to one another (8. vi.):

And because the triangle ABC is similar to ABD; as CB is to BA, so is BA to BD (4. vi.): and because these three straight lines are proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second (2 *Cor.* 20. vi.): therefore as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA: and inversely (*B.* v.), as DB to BC, so is the figure upon BA to that upon BC:

For the same reason, as DC to CB, so is the figure upon CA to that upon CB:



Wherefore as BD and DC together to BC, so are the figures upon BA, AC to that up BC (24. v.):

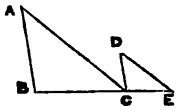
But BD and DC together are equal to BC; therefore the figure described upon BC is equal (*A.* v.) to the similar and similarly described figures on BA, AC. Wherefore, in right-angled triangles, etc. Q. E. D.

PROPOSITION XXXII.

Theor. *If two triangles, which have two sides of the one proportional to two sides of the other, be joined at one angle, so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.*

Let ABC, DCE be two triangles which have the two sides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE: BC and CE are in a straight line.

Because AB is parallel to DC, and the straight line AC meets them, the alternate angles BAC, ACD are equal (29. I.):



For the same reason, the angle CDE is equal to the angle ACD; wherefore also BAC is equal to CDE:

And because the triangles ABC, DCE have one angle at A equal to one at D, and the sides about these angles proportionals, viz. BA to AC, as CD to DE, the triangle ABC is equiangular (6. VI.) to BCE; therefore the angle ABC is equal to the angle DCE:

And the angle BAC was proved to be equal to ACD; therefore the whole angle ACE is equal to the two angles ABC, BAC:

Add the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC, ACB:

But ABC, BAC, ACB are equal to two right angles (32. I.); therefore also the angles ACE, ACB are equal to two right angles: and since at the point C, in the straight line AC, the two straight lines BC, CE, which are on the opposite sides of it, make the adjacent angles ACE, ACB equal to two right angles, therefore (14. I.) BC and CE are in a straight line. Wherefore, if two triangles, etc. Q. E. D.

PROPOSITION XXXIII.

THEOR. *In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another; so also have the sectors.*

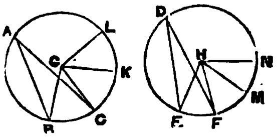
Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences: as the circumference BC to the circumference EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and also the sector BGC to the sector EHF.

Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN, each equal to EF; and join GK, GL, HM, HN.

Because the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are also all equal (27. III.); therefore what multiple soever the circumference BL is of the circumference BC, the same multiple is the angle BGL of the angle BGC:

For the same reason, whatever multiple the circumference EN is of the circumference EF, the same multiple is the angle EHN of the angle EHF:

And if the circumference BL be equal to the circumference EN, the angle BGL is also equal (27. III.) to the angle EHN; and if the circumference BLA be greater than EN, likewise the angle BGL is greater than EHN; and if less, less:



There being then four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF; of the circumference BC, and of the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimultiples whatever, viz. the circumference EN, and the angle EHN; and it has been proved, that if the circumference BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if less, less: as therefore the circumference BC to the circumference EF, so (5 *Def.* v.) is the angle BGC to the angle EHF:

But as the angle BGC is to the angle EHF, so is (15. v.) the angle BAC to the angle EDF, for each is double of each (20. III.); therefore, as the circumference BC is to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.

Also, as the circumference BC to EF, so is the sector BGC to the sector EHF. Join BC, CK, and in the circumferences BC, CK take any points X, O, and join BX, XC, CO, OK:

Then, because in the triangles GBC, GCK, the two sides BG, GC are equal to the two CG, GK, and that they contain equal angles; the base BC is equal (4. I.) to the base CK, and the triangle GBC to the triangle GCK:

And because the circumference BC is equal to the circumference CK, the remaining part of the whole circumference of the circle ABC, is equal to the remaining part of the whole circumference of the same circle: therefore the angle BXC is equal (27. III.) to the angle COK; and the segment BXC is therefore similar to the segment COK (11 *Def.* III.); and they are upon equal straight lines BC, CK:

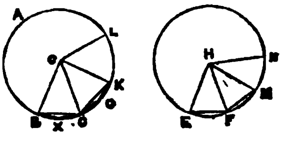
But similar segments of circles upon equal straight lines are equal (24. III.) to one another; therefore the segment BXC is equal to the segment COK:

And the triangle BGC is equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK:

For the same reason, the sector KGL is equal to each of the sectors BGC, CGK:

In the same manner, the sectors EHF, FHM, MHN may be proved equal to one another:

Therefore, what multiple soever the circumference BL is of the circumference BC, the same multiple is the sector BGL of the sector BGC: for the same reason, whatever multiple the circumference EN is of EF, the same multiple is the sector EHN of the sector EHF: and if the circumference BL be equal to EN, the sector BGL is equal to the sector EHN; and if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN; and if less, less:



Since, then, there are four magnitudes, the two circumferences BC, EF, and the two sectors BGC, EHF; and of the circumference BC, and sector BGC, the circumference BL and sector BGL are any equimultiples whatever; and of the circumference EF, and sector EHF, the circumference EN, and sector EHN, are any equimultiples whatever; and that it has been proved, if the circumference BL be greater than EN, the sector BGL is greater than the sector EHN; and if equal, equal; and if less, less: therefore (5 *Def.* v.), as the circumference BC is to the circumference EF, so is the sector BGC to the sector EHF. Wherefore, in equal circles, etc. Q. E. D.

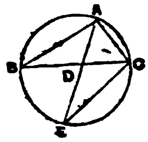
PROPOSITION B.

Theor. *If an angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.*

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD: the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD.

Describe the circle (5. IV.) ACB about the triangle, and produce AD to the circumference in E, and join EC:

Then because the angle BAD is equal to the angle CAE, and the angle ABD to the angle (21. III.) AEC, for they are in the same segment, the triangles ABD, AEC are equiangular to one another:



Therefore as BA to AD, so is (4. VI.) EA to AC; and consequently the rectangle BA, AC is equal (16. VI.) to the rectangle EA, AD; that is (3. II.), to the rectangle ED, DA, together with the square of AD:

But the rectangle ED, DA is equal to the rectangle (35. III.) BD, DC; therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the square of AD. Wherefore, if an angle, etc. Q. E. D.

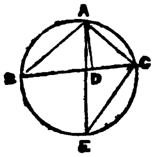
PROPOSITION C.

Theor. *If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

Let ABC be a triangle, and AD the perpendicular from the angle A to the base BC: the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.

Describe (5. IV.) the circle ACB about the triangle, and draw its diameter AE, and join EC:

Because the right angle BDA is equal (31. III.) to the angle ECA in a semicircle, and the angle ABD to the angle AEC in the same segment (21. III.); the triangles ABD, AEC are equiangular:



Therefore as (4. VI.) BA to AD, so is EA to AC; and consequently the rectangle BA, AC is equal (16. VI.) to the rectangle EA, AD. If, therefore, from any angle, etc. Q. E. D.

PROPOSITION D.

Theor. *The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.*

Let ABCD be any quadrilateral inscribed in a circle, and join AC, BD: the rectangle contained by AC, BD is equal to the two rectangles contained by AB, CD, and by AD, BC.*

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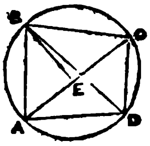
* This is a Lemma of Cl. Ptolomæus, in page 9 of his Μεγάλη Σύνταξις.

Make the angle ABE equal to the angle DBC;

Add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC: and the angle BDA is equal (21. III.) to the angle BCE, because they are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE: wherefore (4. VI.), as BC is to CE, so is BD to DA; and consequently the rectangle BC, AD is equal (16. VI.) to the rectangle BD, CE:



Again, because the angle ABE is equal to the angle DBC, and the angle (21. III.) BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD; as therefore BA to AE, so is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE:

But the rectangle BC, AD has been shown equal to the rectangle BD, CE; therefore the whole rectangle AC, BD (1. II.) is equal to the rectangle AB, DC, together with the rectangle AD, BC. Therefore, the rectangle, etc. Q. E. D.

EXERCISES ON BOOK VI.

1. If perpendiculars be drawn from the angles of a triangle to the opposite sides:—

(*a*) They will be to one another in the reciprocal ratio of the sides upon which they fall:

(*b*) Each pair will cut off a pair of similar right-angled triangles from the original triangle; which are to be specifically assigned, and the statement with respect to them proved.

2. Triangles which have equal or supplementary angles are to one another in a ratio compounded of the ratios of their containing sides.

And conversely, if two triangles be to one another in a ratio compounded of the ratios of their sides, the angles contained by those sides are either equal or supplementary.

Give, in connexion with this, the enunciation of VI. 23, in its most general form.

3. Prove III. 35, 36, 37, by means of similar triangles.

4. Given the ratio of two lines, and either their sum, difference, rectangle, sum of squares, or difference of the squares, to find those lines.

5. Homologous lines will divide similar rectilineal figures into figures which are similar each to its homologous one.

6. Lines drawn homologously (whether from the angular points or otherwise) to homologous points of similar figures, will make the homologous angles equal, each to each; and will have, each to each, the same ratio.

7. If two similar figures have one side parallel to its homologous one, then lines joining the homologous angles intersect in the same point.

[In certain cases, two such points may exist. It will be a good exercise to discover those cases.]

8. Similar rectilineal figures are to one another in the ratio com-

pounded of the ratios of any two lines anyhow drawn in the one to two lines drawn homologously to them in the other.

What does this become when the homologous lines are homologous sides of the figure? And what when both are repetitions of the same side of each figure?

9. Triangles and parallelograms are to one another in a ratio compounded of the ratios of their bases and the ratio of their perpendiculars. Generalise this by taking lines equally inclined to the bases, instead of the perpendiculars.

10. Draw a perpendicular from the right angle to the hypothenuse of a right-angled triangle: then the sides about the right angle are to one another in the sub-duplicate ratio of the adjacent segments of the hypothenuse; and the diameters of the circles inscribed in the two partial triangles are to one another as the adjacent sides of the original triangle.

11. Circles inscribed, escribed, and circumscribed to similar triangles, have their respective diameters in the same ratio that the homologous sides of the triangles have.*

12. Having two lines given, it is required to add to them, to take from them, or to add to one and take from the other, lines in a given ratio, so that the sum, difference, or ratio of the two final lines shall be also given.

13. Through a given point to draw a line terminated by a given circle, such that the sum, difference, or ratio of its segments made at that point shall be given.

14. Given the ratio of the sides and the vertical angle of a triangle, to construct it when the third datum is—

(*a*) The base;
(*b*) The radius of the circumscribed circle;
(*c*) The sum or difference of the sides;
(*d*) The rectangle under the sides;
(*e*) The perpendicular from the vertex to the base;
(*f*) The line from the vertex to the middle of the base;
(*g*) The radius of the inscribed circle;
(*h*) The radius of the circle escribed to the base.

15. Given any three of the radii of the five circles, viz., the circumscribed, the inscribed, and the three escribed, to construct the triangle.

[Many elegant properties are connected with the solution of this problem.]

16. Similar polygons, whether inscribed in or described about circles, have their perimeters in the ratio of the diameters of those circles, and their areas in the duplicate ratio of those diameters; and one inscribed polygon will be to the other in the same ratio as the corresponding circumscribed polygon is to the other.

17. Let two triangles ABC, A'B'C' have their bases AB, A'B' situated in the same indefinite line; and let CC' be drawn to cut this line in D: then will the triangles have to one another a ratio compounded of the ratios of the bases AB : A'B', and of the distances of their vertices from the point of intersection DC : DC'.

---

* A circle which touches one side of a triangle and the other two sides produced is called an *escribed* circle.

Examine in detail the cases:—

(*a*) When D is infinitely remote, or CC' parallel to AB';

(*b*) When the bases AB, $A'B'$ are equal;

(*c*) When the bases are identical also in position;

(*d*) When the vertices are in the same line perpendicular to the base;

(*e*) When the triangles are on different sides of the line in which their bases are situated.

18. Take the figure to I. 47. Let FG, HK be produced to meet in R; and let P, Q be the intersections of FC, KB, with the sides AB, AC respectively, and N that of AL, BC. Then, it is required to prove that—

(*a*) The lines FG, HK intersect in AN produced:

(*b*) The segments AP, AQ are equal; and if lines be drawn from P and Q respectively parallel to AC and AB, they will meet in AL:

(*c*) The points K, A, F are in a straight line.

(*d*) Let AD, AE intersect BC in S, T; and let Z be the intersection of CF, BK in AN. Then

$BS : ST :: ST : TC$; or $BS.TC = ST^2$:
$SN : NZ :: NZ : NT$; or $SN.NT = NZ^2$:
$BS : AZ :: AZ : CT$; or $BS.CT = AZ^2$.

Properties of this figure appear to be inexhaustible. Upwards of fifty are given by Mr. Bransby in Vol. III. of the "Math. Repos.;" and as geometrical exercises in earlier life, I once deduced about ten times that number. They appeared at the end to be more plentiful than at the beginning.

19. Turn to the figure in IV. 10, and show that

(*a*) $AC^2 - CB^2 = AC.CB$:

(*b*) $AC : CB :: \sqrt{5} - 1 : 3 - \sqrt{5}$:

GEOMETRY OF PLANES AND SOLIDS.

PRELIMINARY NOTE.

Two or three cautions must be given at the outset.

1. This subject is not to be considered difficult or abstruse, from its being so long delayed in its formal introduction into a geometrical course. The delay has arisen from the system of keeping the subjects of plane and solid geometry separate, and of discussing the one fully before the other is entered upon. There would, indeed, be no impropriety in the actual study of the two branches of the subject being carried on *pari passû;* although such a practice is not customary, and may have, if adopted, no especial advantage.

The only real difficulty that is attached to this part of the subject arises from the use of pictures of the figures instead of models of them *in books*. The picture in a plane proposition has a similitude in its general features to the perfect one which the mind contemplates in its investigations; but this is lost when we use pictorial representations of figures in space, and the imagination is taxed to consider things as equal which are incapable in the picture of approximating to equality, and other such relations. However, if we substitute a model (no matter how rudely formed) for the picture, the difficulty is at once and entirely removed.

Beyond this there exists not the least difficulty in the conception or complexity in the reasoning, that can render the discussion of the line and plane, and figures formed by them, more intricate to the student than the First Book of Euclid was: indeed, from the power created by his previous geometrical experience, it ought to be less so to his mind. The propositions of plane geometry which are brought into play in the solid geometry are, moreover, of the most simple and elementary kind—mostly the First Book of Euclid,—the others but rarely.

2. The student must take care to get clear views of the geometrical *assumptions* respecting the line and plane. A few remarks on this subject are subjoined.

> (*a*.) We have a distinct conception of the line and plane antecedently to all *geometrical argument*, or even of *verbal definition*. No form of words can render one or the other more clear to the mind; but, on the contrary, all attempts at verbal definition of them tends to confuse the understanding. We do not inquire how the mind *forms* these conceptions, or whether they be innate or not: it is sufficient that they are uniform and universal.

Besides the conception of the figures, we have equally fixed in the mind certain relations of them. These are—that two straight lines cannot coincide in two points without coinciding altogether to the utmost extent of their common prolongation; that a straight line cannot coincide with a plane in two points without also coinciding with it at every point in that line; and that a plane cannot coincide with a plane in three points, not in a right line, without the coincidence being entire to the utmost common extension of those planes.

(*b.*) From these properties, which are properly called "axioms," other axioms (which partake of the nature of corollaries) are deducible. For instance, innumerable planes may pass through the same straight line, or through the same two points. Two planes which meet one another intersect in a straight line. A plane may be conceived to revolve about a given straight line till it passes through any given point without that line.*

(*c.*) Let it be clearly understood, that an axiom declares a property of which no *logical proof* is offered. The ground upon which it rests is antecedent to the logical part of geometry; its character and evidence is a question of metaphysics, and the axiom must be admitted before reasoning in a logical sense can be commenced. Let the student above all things keep clear of the frequent delusion of its being possible to build up a system of "geometry without axioms," whether the basis be definitions or whatever else.

3. Let it also be understood what is the proper and true character of a definition. It is always a description of the thing meant by a word. The *mode* of description might be varied; but a definition is always, as has been just stated, nothing more and nothing less. The statement always describes the conditions requisite for making the figure, as Euclid's definitions of a triangle or a circle. They are often given in those absolute forms; but almost as often by describing the *operations* which will produce the figure. Thus a circle may be defined: "If one extremity of a given straight line be fixed, and the line revolve about it in a plane, the other extremity will describe a circle." Euclid's definition of a sphere is:—"A sphere is a solid figure described by the revolution of a semicircle about its diameter which remains fixed." This might have been given thus: "A sphere is a figure contained by one surface, and is such that every point in that surface is equally distant from a point within it." It thus appears to be immaterial in some cases whether the figure be defined as previously existing or by means of its genesis. On the whole, it conduces to simplicity and brevity to define figures bounded by straight lines and planes in an absolute manner, and those bounded (or partially bounded) by curves and curve surfaces by means of their genesis. We are not, however, imperatively tied down to this arrangement; but whatever method we employ, the description must be adequate, complete, and devoid of superfluous conditions.

The postulate simply predicates that the elementary figures are constructible, leaving out all implication of the means of making them, as in the three in Euclid's First Book. The cause appears to be,—not that "the process is mechanical," or that "the method is self-evident," or any one of the many conjectures that have been made; but that if the *modus operandi* were predicated, no means exist for demonstrating the truth of the construction. The *form of the problem* could hardly be given to them under these circumstances, as their being left without demonstration would *in initio* violate the general rule upon which the system was developed.

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* To these may be added the inferences:—the three sides of a triangle are in one plane; if two straight lines meet one another, they are in one plane; and some others of like kind, which are occasionally required, but which do not require to be dwelt upon at length here.

4. The construction of the problems in space that are requisite as subordinate to demonstration are always given by Euclid; and, indeed, an adherence to this practice is in some respects desirable. Those which occur are however so simple, and the processes little more than corollaries from theorems actually given, that I have yielded to the wishes of the officer to whom the Board has committed the control of these volumes, and omitted them altogether. A few of them will, however, inevitably make their appearance in the "Descriptive Geometry" hereafter.

DEFINITIONS.

1. *Parallel planes* are such as, however far produced in all possible directions, they will never meet.

2. *A line and plane are parallel* when, however far produced in all possible directions, they will never meet.

3. *A dihedral angle* is formed by two planes which intersect. The line of intersection is called *the edge of the angle*, and the planes themselves *the faces of the angle.*

4. If from any point in the edge of a dihedral angle two straight lines be drawn, one in each of the faces perpendicular to the edge, they form the *profile angle of the planes.* Their inclination to one another is called the *inclination of the planes.* If the profile angle be a right angle, the planes are said to be *perpendicular* to one another.

5. If a straight line be perpendicular to each of two straight lines at their point of intersection, it is said to be *perpendicular to the plane* passing through those two lines. Conversely, the plane is said to be *perpendicular to the line.*

6. A line which is neither parallel nor perpendicular to a plane is said to be *oblique* to the plane. *The inclination of an oblique line to a plane* is the acute angle contained by that line and another drawn from the point in which it meets the plane, to the point in which a perpendicular from any point in the line meets the plane.

7. *The trace of a line or plane* (or any surface, indeed) is the point or line in which that line or plane meets some specified plane.

8. *A solid angle* (or more descriptively, a *polyhedral angle*), is formed by three or more planes which meet in a point, and each plane limited in its expansion by the two adjacent ones.

It is called *trihedral*, *tetrahedral*, *pentahedral*, etc., according as there are three, four, five, etc., planes.

The planes are called *faces;* their linear intersections, *edges;* and the point in which they all meet, the *vertex* of the solid angle.

9. If any number of lines be parallel and intercepted between two parallel planes, and planes join these two and two consecutively, the figure produced is a *prism.*

The prism is, for geometrical purposes, considered as capable of indefinite prolongation both ways from the parallel planes, or in the direction of the lines.

The lines are called the *edges* of the prism; the parallel planes, the *ends* or *bases* of the prism; and the planes joining the parallel lines, the *faces.*

10. If planes passing through a point without a plane, pass also through the sides of a polygon in that plane, and be limited by their adjacent intersections, they form a *pyramid.*

The pyramid takes the name of *triangular*, *tetrangular*, *pentangular*, etc., according to the figure of the polygon.

That polygon is the *base*, the point is the *vertex*, the planes are the *faces*, and their intersections the *edges* of the pyramid.

The pyramid is, geometrically, capable of indefinite extension, both below the base and beyond the vertex.

11. If a line making any angle with the plane of a circle move continually over the circumference, and always keep parallel to its first position, it generates a *cylindrical surface.*

If this surface be cut by two planes parallel to the circle, the intercepted portion is a *cylinder.*

These two planes are called the *ends* or *bases* of the cylinder; the original circle the *directrix;* the moving line the *generatrix;* and a line through the centre parallel to the generatrix, the *axis* of the cylinder.

12. If about a point taken without the plane of a circle a straight line move always touching the circumference of the circle, it generates a *conical surface.* The portion of it intercepted between the point and circle is called a *cone.* The point is the *vertex* of the cone, the circle is its *base*, and the generatrix in any position is called an *edge* of the cone.*

13. If a circle revolve about any diameter till its plane takes a reversed position, it will generate a *spherical surface* or a *sphere.*

The terms *centre*, *diameter*, etc., are the same as in a circle.

14. *Polyhedron* is a figure bounded by plane faces, and takes its name from the number of those faces.

Thus the *tetrahedron* has four faces, the *hexahedron* six, the *octohedron* eight, the *dodecahedron* twelve, and the *icosahedron* twenty. These names, however, are only applied when all the faces of the figure are equal to one another, forming subjects of much interest to the ancients; but, except from their being amongst the most frequent forms assumed by crystals (both artificially and naturally produced), they would be mere curiosities in modern science.

In modern science the polyhedron is always described, but seldom named, with one or two exceptions.

15. When three pairs of parallel planes mutually intersect, they enclose a figure called the *parallelopiped.*

It is *oblique* or *rectangular*, according as the containing pairs of planes are oblique or rectangular. The *cube* is the most confined instance of it.

16. *Similar polyhedrons* are such as have their angular points or vertices similarly situated, each with respect to the others.†

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* The terms cone and cylinder are generally employed to signify not only the parts here defined by them, but the surfaces generally extended as far as may be necessary.

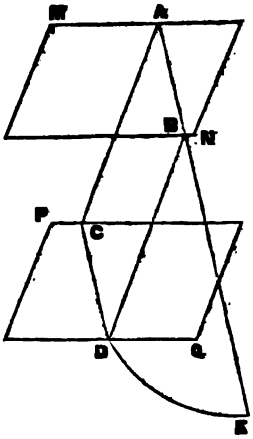
† This definition is complete; all others yet given involve superfluous conditions, just as the definition of similar plane polygons does.

CHAPTER I.

PARALLELISM OF LINES AND PLANES.

PROPOSITION I.

If two parallel planes be cut by a third plane, the sections will be parallel.



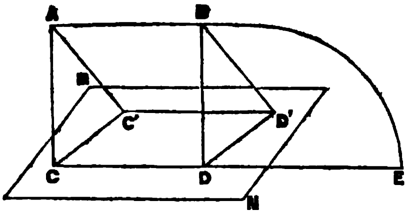
Let the parallel planes MN, PQ be cut by the plane AD in the lines AB, CD: these lines, AB, CD, will be parallel.

For since AB, CD are in one plane, if they be not parallel, they will meet in some point, as E.

Then since AB is in the plane MN, the point E is in that plane; and similarly, E being in the line CD, it is also in the plane PQ. The planes MN, PQ therefore meet in E, which is impossible (*Def.* 1.) since they are by hypothesis parallel to one another.

PROPOSITION II.

If a straight line be parallel to a plane, all planes through this line which cut that plane will have their sections with it parallel to the line parallel to it, and to one another.



Let the line AB be parallel to the plane MN, and the planes AD, AD' drawn through AB, cut the plane MN in CD, C'D': then CD, C'D' will be parallel to AB, and to one another.

(1.) Since AB, CD are in one plane, they are either parallel to one another, or they will meet in some point E.

If parallel, the proposition is admitted; but if not, let AB, CD meet in some point E. Then since E is in the line CD, which is itself in the plane MN, E is in the plane MN; and the line AB meets the plane MN in the point E. But this is impossible, since AB is parallel to

MN by the hypothesis. Wherefore since AB, CD are in one plane ABCD and never meet, they are parallel. Similarly, C'D' is parallel to AB.

(2.) The planes AD, AD' intersecting in the line AB, they cannot meet in any point out of that line; and hence the lines CD, C'D' in them, and parallel to AB, can never meet: but they are in one plane MN; and hence they are parallel.

PROPOSITION III.

If a line without a plane be parallel to a line in the plane, the first line will be parallel to the plane.

(Preceding figure.)

Let the line AB without the plane MN be parallel to the line CD in it: then AB will be parallel to the plane MN.

For since AB, CD are parallel, they are in one plane, and can never meet; and since AD and MN meet in the line CD, they cannot meet in any point without CD. Whence AB, lying wholly in the plane ACD, can never meet the plane MN; that is, AB is parallel to MN.

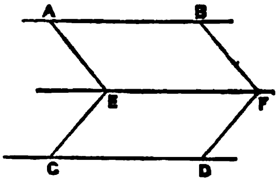
Cor. If through CD one of two parallel lines AB, CD, a plane be drawn; this plane will either wholly include AB, or be parallel to it.

PROPOSITION IV.

If through two parallel lines planes be drawn, they will either coalesce, be parallel, or have their intersection parallel to those two lines.

(1.) The planes *may* each be drawn through the other line, since parallel lines are in one plane; and hence they may coalesce with the plane which contains the parallels, and therefore with one another.

(2.) They *may* never meet, since the lines through which they pass never meet; and in this case they would be parallel.



(3.) Let AB, CD be parallel lines, through which are drawn any two planes AF, ED which neither coalesce, nor are parallel: they will meet in some straight line EF; and this line is parallel to each of the lines AB, CD.

For since AB is parallel to the line CD lying in the plane ED, it is parallel to the plane ED itself (*Prop.* III.); and hence it can never meet the line EF in that plane. But EF is also in the plane AF, since it is the intersection of that plane with the plane ED; and hence AB, EF which lie in the same plane and never meet, are parallel.

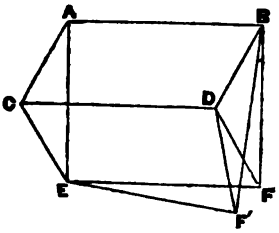
In the same way it is proved that EF is parallel to CD.

PROPOSITION V.

If two planes intersect, and lines be drawn in those planes parallel to the intersection of the planes, one in each, these lines will be parallel.

Let the two planes AD, DE intersect in CD; and in them respectively let AB, EF be drawn parallel to CD: then AB, EF will be parallel to one another.

(1.) If AB, EF be not in one plane, draw a plane through AB and E to cut the plane CF in EF′. Then since through AB and CD the planes AF′, CF′ are drawn, intersecting in EF′, the line EF′ is parallel to CD (*Prop.* IV.). But (*Hypoth.*) EF is parallel to CD; and hence through the point E two lines EF, EF′ are drawn parallel to CD, which is impossible.



Wherefore AB, EF are in one plane.

Again, since CD is parallel to the line AB in the plane ABEF, it is parallel to the plane itself (*Prop.* III.); and since through the line CD which is parallel to the plane ABEF planes AD, DE are drawn to cut it in the lines AB, EF, these lines are themselves parallel to one another (*Prop.* II.).

PROPOSITION VI.

If each of two lines be parallel to a third line, they will be parallel to one another.

(Same figure.)

Let AB and EF be each parallel to CD; they will be parallel to one another.

For since AB, CD are parallel, they are in one plane; and similarly EF, CD are in one plane. These planes intersect in CD; and parallel to CD the lines AB, EF are drawn in those planes respectively. Wherefore AB, EF are parallel (*Prop.* V.).

Cor. 1. If any number of lines be each of them parallel to one line, they will be parallel each to all the others.

Cor. 2. If any number of lines be parallel to one of two parallel lines, and any number parallel to the other, all the lines will be parallel, each to all the others.

For the proof may be extended to a fourth, a fifth, etc., parallel line without limitation as to number. Let the student apply it to an instance or two in detail.

PROPOSITION VII.

If two straight lines which meet be parallel to two others which also meet, but not in the same plane with them: then

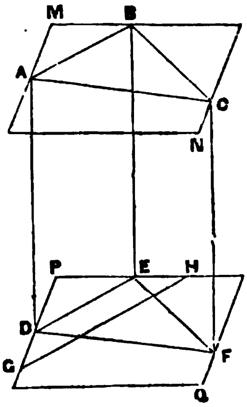
(1.) *The plane which contains the first two will be parallel to that which contains the other two; and*

(2.) *The angle contained by the first two will be equal to the angle contained by the two others.*

(1.) Let the lines AB, BC be respectively parallel to DE, EF: then the plane MN through the former pair will be parallel to the plane PQ through the latter pair.

For if not, let the plane MN cut PQ in some line, as GH. Then, since AB is parallel to a line DE in the plane PQ, it is parallel to PQ (*Prop.* III.); and since the plane MN is drawn through the line AB, which is parallel to PQ to meet PQ in GH, GH is parallel to AB (*Prop.* II.); and therefore, again, to DE (*Prop.* VI.).

In the same manner BC is parallel to the plane PQ; and the plane MN cuts the plane PQ in a line GH parallel to BC, and therefore to EF. The line GH is therefore parallel to each of two straight lines DE, EF which meet; but this is impossible, and hence the planes MN, PQ cannot meet, that is, they are parallel.



(2.) The angle ABC is equal to the angle DEF. Take AB, BC equal respectively to DE, EF, and join AC, DF, AD, BE, CF.

Then since AB is equal and parallel to DE, the figure ABED is a parallelogram, and the opposite sides AD, BE are equal and parallel (*Euc.* 33, I.). Similarly CF is equal and parallel to BE. Whence (*Prop.* VI.) AD is equal and parallel to CF; and therefore, again, AC to DF. The conclusion hence follows from *Euc.* 8, I.

PROPOSITION VIII.

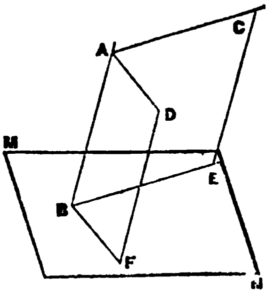
Let a plane and a point without it be given: then

(1.) *Innumerable straight lines can be drawn through the point, parallel to the plane;*

(2.) *Only one plane can be drawn through the point parallel to the plane;*

(3.) *All these lines will be in one plane parallel to the given plane.*

(1.) Let A be the given point, and MN the given plane: innumerable lines AC, AD, etc. can be drawn through A parallel to the plane MN.



For take any point B in MN and join AB; draw any lines BE, BF, etc., in the plane MN; draw planes through AB and BE, AB and BF, etc.; and in these planes draw AC parallel to BE, AD to BF, etc.

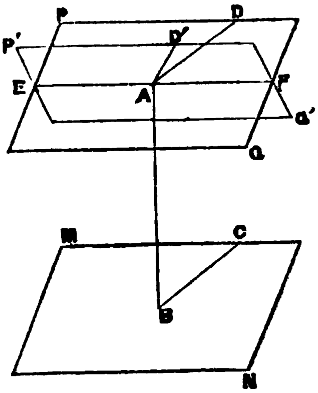
Then, since AC without the plane MN is parallel to the line BE in that plane, the line AC is parallel to the plane MN. In the same manner AD is parallel to MN; and so on with the innumerable lines which may be drawn in the same manner through A.

(2.) There can be but one plane drawn through A parallel to MN.

For, if possible, let two planes PQ, P′Q′ both passing through A be parallel to MN; and let EF be their intersection. Through AB (taken as previously) draw any plane not coinciding with EF, and let it cut MN in BC, PQ in AD, and P′Q′ in AD′.

Then, since the planes PQ, P′Q′ intersect in EF, they can have no other point common than in EF, and AD, AD′ do not coincide in any other point than A.

Also since the plane ABC meets the parallel planes MN, PQ, the lines BC and AD are parallel (*Prop.* I.). And since it meets the parallel planes MN, P′Q′ the lines BC, AD′ are parallel. Whence through the point A two lines AD, AD′ are drawn parallel to the same line BC in the plane ABC, which is impossible; wherefore only one plane can be drawn through A parallel to MN.



(3.) All lines drawn through A parallel to the plane MN lie in the plane PQ parallel to MN.

For if it be possible, let AD′ not in the plane PQ be parallel to MN. Take AB as before, and through D′A, AB, draw the plane D′ABC, meeting MN in BC and PQ in AD.

Then, since AD′ is parallel to the plane MN, it is parallel to BC (*Prop.* II.); and since the plane D′A BC meets the parallel planes PQ, MN in AD, BC, the line AD is parallel to BC (*Prop.* I.). Whence in the plane D′ABC there are drawn through the same point A two lines AD, AD′ parallel to BC, which is impossible. The line AD′ cannot therefore be at the same time parallel to MN and without the plane PQ.

Cor. 1. Any line drawn in one of two parallel planes is parallel to the other plane.

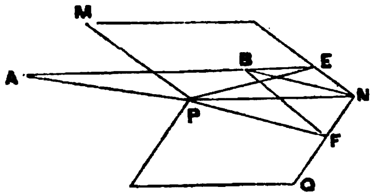
Cor. 2. If a straight line be parallel to a plane, a plane parallel to that plane can always be drawn through the line.

The application of the preceding reasoning to this corollary is left for the student to make.

PROPOSITION IX.

(1.) *If a straight line be parallel to the intersection of two planes, it will be parallel to each of those planes; and*

(2.) *If a straight line be parallel to each of two planes which intersect, it will be parallel to their intersection.*



(1.) If a straight line AB be parallel to the intersection PN of two planes MN, PQ, it will be parallel to the planes MN, PQ.

For since AB is parallel to the line PN in the plane MN, it is parallel to the plane MN itself (*Prop.* III.). In the same way it is shown to be parallel to PQ.

(2.) If AB be parallel to each of the planes MN, PQ, it will be parallel to their intersection PN.

Let a plane pass through AB and P, a point in PN, then it may be supposed to coincide with PN or not to coincide.

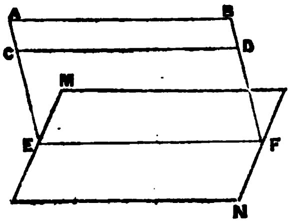
In the first case, since AB is parallel to the planes MN, PQ, it can never meet either of them; nor, consequently, can it meet PN, their common section. But AB, PN being in one plane, and never meeting, they are parallel.

In the second case, let the plane ABP not coincide with PN, but cut the planes MN, PQ in some other lines PE, PF.

Then since AB is parallel to MN, it is parallel to PE, the intersection of the plane ABP with MN (*Prop.* II.); and, similarly, AB is parallel to PF. Whence the two lines PE, PF in the plane ABP are parallel to the same line AB, which is impossible. Whence AB is parallel to the intersection PN of the planes MN, PQ.

PROPOSITION X.

If a plane be parallel to one of two parallel lines, or to one of two parallel planes, it will be parallel to the other.

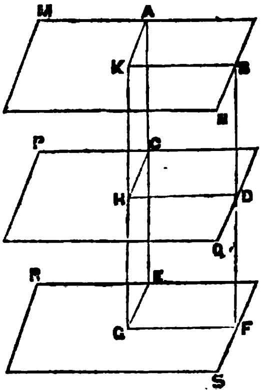


(1.) Let the plane MN be parallel to AB, one of the two parallel lines AB, CD, it will be parallel to the other.

For since AB, CD are parallel, they are in one plane; and the plane ABCD will either be parallel to MN or intersect it in some line EF.

If the plane MN be parallel to ABCD, it is parallel to every line AB, CD, etc. in ABCD (*Prop.* VIII. *Cor.* 1.).

Let, on the other hand, the planes ABCD and MN intersect in EF; then, since MN is parallel to AB, the line EF is parallel to AB (*Prop.* II.). Also, since CD is parallel to AB (*Hypoth.*) and AB to EF, the line CD is parallel to a line EF in the plane MN; and hence the line CD and plane MN are parallel (*Prop.* III.).



(2.) Let the plane RS be parallel to one of the parallel planes MN, PQ, as MN: it will be parallel to the other, PQ.

In the extreme planes, MN, RS take points K, G, and join KG. This will necessarily cut PQ in some point H.

Through KHG draw any two planes cutting the three planes in KA, HC, GE and KB, HD, GF, respectively.

Then, since MN is parallel to RS, the section AK is parallel to GE (*Prop* I.); and since MN is parallel to PQ, the section AK is parallel to CH. Whence the lines EG, CH in the plane KE are parallel to AK, and hence they are parallel to one another.

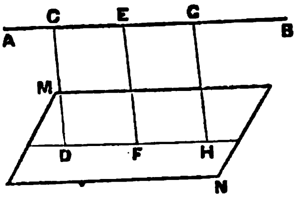
In the same manner it is shown that GF is parallel to KB.

Wherefore the lines AK, KB being parallel to EG, GF, the planes RS, MN drawn through them are parallel (*Prop.* VII.).

PROPOSITION XI.

If a line be parallel to a plane, and from points in the line parallel lines be drawn to meet the plane, these will all be situated in one plane, and equal to one another.

Let AB be parallel to the plane MN, and let parallel lines CD, EF, GH, etc. be drawn from points C, E, G, etc. in AB to meet the plane: then CD = EF = GH = etc; and all these lines will be situated in one plane.

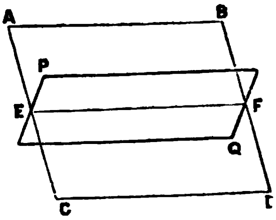


(1.) For, since CD, EF are parallel, they are in one plane; and the points C, E being in that plane, the line EC which joins them, and its extension AB are in that plane. In like manner all the others, GH, etc. will be in that plane; viz., the plane which passes through AB and any one of them CD.

(2.) Also since this plane cuts MN in a line DH parallel to AB (*Prop.* II.), the figure CDFE is a parallelogram, and hence the opposite sides CD, EF are equal. Similarly all the others, GH, etc. are equal to CD.

PROPOSITION XII.

If one of two parallel lines be parallel to a plane, the other is also parallel to it.



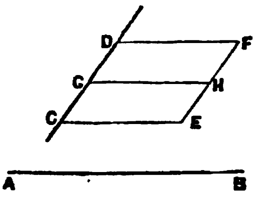
Let AB, CD be parallels, and one of them AB be parallel to the plane PQ; then the other CD will also be parallel to PQ.

For, since AB and CD are parallel, they are in one plane. Let that plane cut PQ in EF, then EF is parallel to AB (*Prop.* II.); and EF being drawn in the plane AD of the two parallels, parallel to one of them, it is parallel to the other CD. Whence CD without the plane PQ is parallel to a line EF in the plane, and is therefore parallel to the plane PQ itself (*Prop.* III.).

PROPOSITION XIII.

If through points in one straight line parallels to another straight line be drawn, these parallels will all be in the same plane, and that plane will be parallel to the same line that the parallel lines are.

Through points C, D, etc. in the line CD let parallels CE, DF, etc. be drawn to the line AB: then all these parallels will be in one plane, and that plane will be parallel to AB.

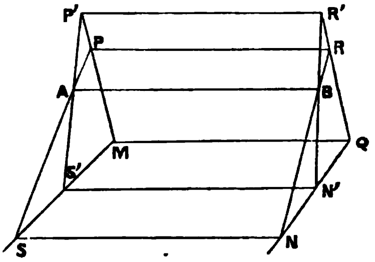


For since CE, DF are both parallel to AB, they are parallel to one another (*Prop.* VI.); and hence they are in the same plane CF; and the points C and D in them are also in that plane. But C, D being in the plane CF, the line CD is in that plane; and hence the second, third, etc., parallel to CE are in the plane which contains DC, CE: that is, all the parallels to AB passing through points in CD are in the one plane DE or CF.

Also since AB is parallel to a line CE in the plane CF, it is parallel to the plane itself (*Prop.* III.): that is, the plane CF is parallel to the line AB.

PROPOSITION XIV.

If a line be parallel to each of two planes, then every plane drawn through this line to cut them, will cut them in parallel lines.



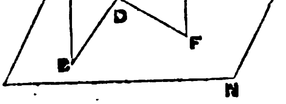
Let the line AB be parallel to each of the planes MN, PQ; and any plane RS be drawn through AB to cut them in PR, SN; then PR will be parallel to SN.

For since AB is parallel to both the planes PQ, MN, it is parallel to their common section MQ (*Prop.* IX.). Whence MQ a line without the plane RS is parallel to the line AB in it; and MQ therefore is parallel to the plane RS (*Prop.* III.).

Then, again, since through the line MQ parallel to the plane RS the planes MN, PQ are drawn to cut RS, the sections NS, PR are parallel (*Prop.* II.).

PROPOSITION XV.

Parallel lines intercepted between parallel planes are equal.



Let AB, CD, EF, etc., be parallel to one another, and limited by the parallel planes MN, PQ; they will be all equal.

For, join AC and BD, CE and DF, etc. Then, since AB, CD are parallel, they are in one plane; and the intersections AC, BD of this plane with PQ, MN are parallel (*Prop.* I.). Wherefore ACDB is a parallelogram, and the opposite sides AB, CD are equal.

In the same way CD and EF are proved equal; and so on however many lines there be.

PROPOSITION XVI.

If two planes which meet one another be parallel to two other planes which meet one another, the intersection of the first two will be parallel to that of the other two.

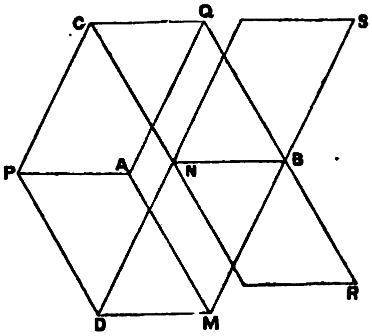
Let the planes PQ, PM which meet in AP be parallel to NS, NR which meet in NB: then AP will be parallel to NB.

For, produce the plane RN to meet the plane PQ in CQ.

Then since the parallel planes PQ, NS are cut by the plane CR, the sections BN and CQ are parallel (*Prop.* I.).

Also, since the parallel planes CR, PM are cut by the plane PQ, the section AP is parallel to the section CQ.

But CQ is also parallel to NB: whence AP is parallel to NB (*Prop.* VI.).



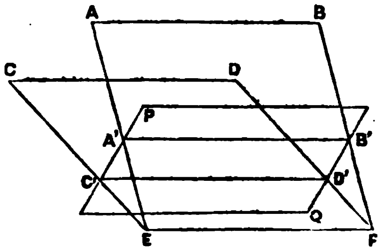
COR. If two planes which are parallel meet two others which are parallel, the four lines of intersection will be parallel. Thus, the four planes PQ, PM, CR, DS meet in the four parallel lines CQ, AP, DM, BN.

PROPOSITION XVII.

If through two parallel lines any planes whatever be drawn to meet a plane parallel to one of the lines, the sections with that plane will be parallel to each other and to the first two lines.

Let AB, CD be two parallel lines and PQ a plane parallel to one of them; and through AB, CD let any planes be drawn to cut PQ in A'B', C'D': these lines will be parallel.

For produce the planes AB', CD' to meet in EF. Then since AB, CD are parallel, the line EF is parallel to them (*Prop.* IV.).



Also since PQ is parallel to one of the parallel lines, AB, it is parallel to the other EF; and hence the planes through EF (that is, through AB, CD) cut PQ in lines A'B', C'D' which are parallel to EF and hence to AB, CD.

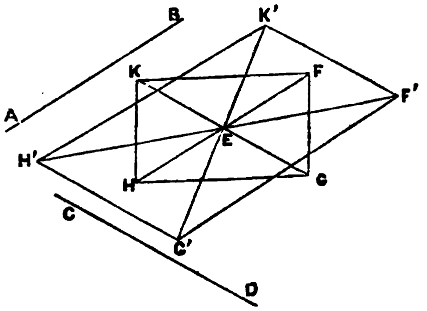
PROPOSITION XVIII.

Through a given point, only one plane can be drawn parallel to two given lines, except those lines be parallel; but when they are parallel, innumerable planes can be drawn parallel to both of them.

(1.) Let AB, CD be two lines not parallel (but in the same plane or not) and E be a given point: then one plane can be drawn through E parallel to both the lines AB, CD.

Through E draw a line HF parallel to AB, and a line KG parallel to CD; and through HF and KG, the plane HKFG.

Then since HF in the plane HKFG is parallel to the line AB, the plane itself is parallel to AB (*Prop.* III.); and similarly, the plane HKFG is parallel to CD. Whence one plane can be drawn as enunciated.



(2.) There can be *only one* plane drawn through E parallel to both the lines AB, CD.

For if there can be a second plane let it be H'K'F'G'; through AB and E draw a plane to cut it in H'F'; and again through CD and E a plane to cut it in K'G'.

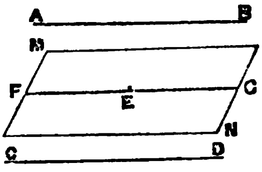
Then H'F' is parallel to AB (*Prop.* II.); and likewise (*by Constr.*) HF is parallel to AB: whence through the same point E two lines HF, H'F' have been drawn parallel to the same line AB; which is impossible.

The same, were it necessary, might be shown by means of KG, K'G'.

One plane only, therefore, can be drawn through E parallel to AB and CD, these lines not being parallel to one another.

(3.) But let AB, CD be parallel, then innumerable planes may be drawn through E parallel to both of them.

For through E draw the line FG parallel to AB or CD, and it will be parallel to the other (*Prop.* VI.); then through FG draw any plane whatever, MN.



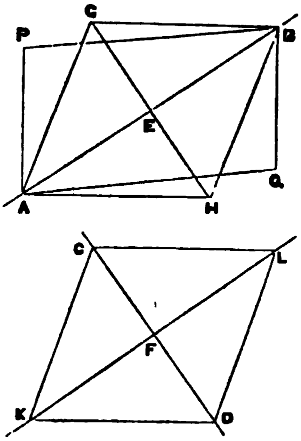
Then since the line AB without the plane MN is parallel to FG in the plane, the line AB is parallel to the plane MN (*Prop.* III.). In the same way CD is parallel to the plane MN. The plane MN is therefore parallel to both the parallel lines AB, CD; and is any whatever subject only to the condition of passing through FG. The number of such planes, then, is unlimited, as affirmed in the proposition.

PROPOSITION XIX.

Through two given lines, not in the same plane, one pair of parallel planes can be drawn, one plane through one line and the other through the other: and only one such pair of parallels can be drawn.

(1.) Let AB, CD be two lines not parallel, then one pair of planes AGBH, KCLD can be drawn, the first through AB and the second through CD, which shall be parallel to one another.

For take any points E, F in AB and CD; through E draw GH parallel to CD, and through F draw KL parallel to AB. Then (*Prop.* VII.) the plane through AB and GH will be parallel to the plane through CD and KL. One such pair, therefore, can be drawn.



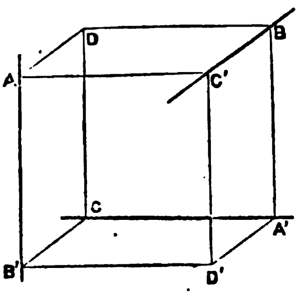
(2.) There cannot be a second pair of such parallel planes.

For if there can, each plane must also be parallel to the other line (*Prop.* VIII. *Cor.* I.); let us admit, then, that a second plane PQ can be drawn through AB which shall be parallel to CD.

Then, since CD is parallel to the two planes PQ and AGBH, it is parallel to their intersection AB (*Prop.* IX.) and hence, AB, CD are in one plane. But by hypothesis they are not in one plane, which is impossible. Whence only one such pair of planes can be drawn.

PROPOSITION XX.

One parallelopiped, and only one, can always be constituted, which shall have three of its non-contiguous edges coincident with three given lines; provided that no two of these lines be in one plane, nor all three parallel to one plane.



(1.) Let AB′, BC′, CA′ be three straight lines, no two of which are in one plane, nor all three parallel to one plane; then a parallelopiped DD′ may be constituted so as to have three non-contiguous edges coincident with these lines.

For through AB′, planes B′D, AD′ can be drawn respectively parallel to the lines BC′, CA′; through BC′, planes C′D, BD′ parallel to the lines CA′, AB′, and through CA′, the planes CD′, A′D parallel to BC′, AB′.

Then the planes B′D and BD′ will be parallel, C′D and CD′ will be parallel, and A′D and AD′ will we parallel (*Prop.* XIX.). These three pairs of parallel planes, when limited by their intersections, constitute a parallelopiped, the non-contiguous edges of which coincide with the three lines AB′, BC′, CA′.

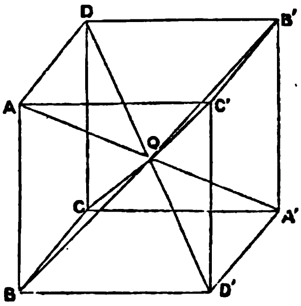
(2.) There cannot be a second parallelopiped, since there cannot be a second pair of component parallel planes drawn through either pair of lines (*Prop.* XIX.).

PROPOSITION XXI.

The six faces of a parallelopiped are parallelograms; those which are opposite are equal: the edges are equal and parallel in fours; and the diagonals which join the four pairs of opposite angles meet in one point, and are bisected there.

(1.) Let the parallelopiped have for its opposite planes AC and A′C′, AB′ and A′B, BC′ and B′C: then these figures are parallelograms.

For since the parallel planes AC, A'C' are cut by the parallel planes AB', A'B, the four lines of section AD, BC, B'C', A'D' are all parallel (*Prop.* xvi. *Cor.*).



In the same manner it is shown that AC', DB', A'C, D'B are parallel; and AB, CD, A'B', C'D' are parallel.

Wherefore AD being parallel to BC, and AB to CD, the figure ABCD (one of the faces of the parallelopiped) is a parallelogram: and in the same way it is proved that all the others are parallelograms.

(2.) The opposite pairs of parallelograms, as AC, A'C', are equal.

For AD, BC, A'D', B'C' being parallel and intercepted between parallel planes AD', A'D, they are equal (*Prop.* xv.). In the same manner AB, CD, A'B', C'D' are equal; and likewise AC', B'D, BD', A'C are equal.

And since AB, BC are parallel to A'B', B'C', the angle ABC is equal to the angle A'B'C'; and parallelograms which have equal angles and the sides about those angles equal each to each, are wholly equal. Whence the parallelograms AC, A'C' are equal.

In the same way is proved the equality of the parallelograms AB' and A'B, and of AD' and A'D.

(3.) The opposite edges are equal in fours. This has been proved in the demonstration of the preceding case.

(4.) The four diagonals AA', BB', CC', DD' intersect in one point Q, and Q is the middle of each of them.

For, if AB', A'B were drawn, the figure AB'A'B would be a parallelogram, of which the diagonals are AA' and BB'; and Q their intersection is the middle of each of them.

In the same manner if AD', A'D were drawn, ADA'D' would be a parallelogram, of which the diagonals are AA', DD'; and the diagonal DD' therefore passes through Q the middle of AA', and is bisected there.

In a similar manner CC' passes through Q, and is bisected there.

Whence all the four pass through one point, and are bisected in that point.

PROPOSITION XXII.

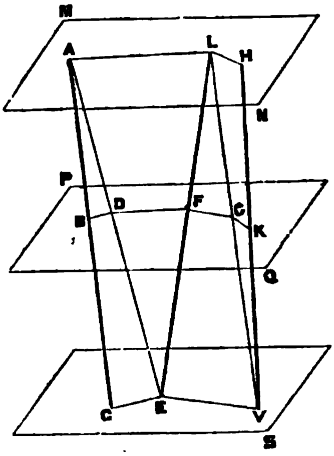
If any number of lines, anyhow situated, be all cut by any number of parallel planes, the planes will divide the lines proportionally.

Let, for instance, the three lines AC, LE, HV, be cut by the three parallel planes MN, PQ, RS, in A, L, H, in B, F, K, and in C, E, V respectively: then they will be divided in these points, so that

$$AB : AC :: LF : LE :: HK : HV.$$

For, draw the planes ACE, AEL, ELV, VLH. These will cut each other, the two former in the line AE, and the two latter in the line LV. They will also cut the parallel planes in parallel lines (*Prop.* I.); viz., so that BD is parallel to CE, DF to AL, FG to EV, and GK to LH. Whence the pairs of triangles ABD and ACE, etc., are similar; consequently,

$$AB : AC :: AD : AE :: LF : LE :: LG : LV :: HK : HV;$$



where we have the enunciated property.

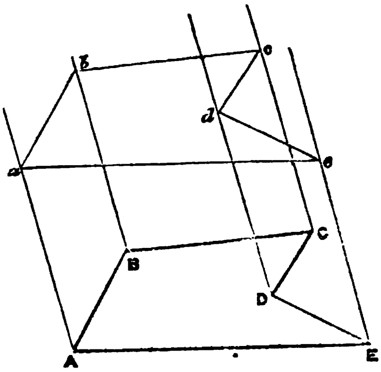
In the same way, the demonstration may be extended to any number of lines and any number of planes.

Cor. 1. Conversely, if any number of lines, more than two, between parallel planes be similarly divided, the sets of homologous points of division will always be situated in so many planes parallel to the original ones.

Cor. 2. When there are only two such lines divided, planes through those pairs of points may be drawn parallel to the original planes; but as two points do not fix the position of a plane, there may be innumerable planes drawn through the pairs of homologous points which are neither parallel to the original planes nor to one another.

PROPOSITION XXIII.

Any section of a prism or pyramid parallel to the base is similar to the base; and that of the prism also equal to the base.



(1.) Let ABCDE be the base of a prism, and the prism itself be cut by a plane parallel thereto in *abcde*: then the figure *abcde* will be similar and equal to ABCDE.

For since the plane AB*ba* cuts the two parallel planes ABCDE, *abcde*, the lines AB, *ab* are parallel (*Prop.* I.). Similarly BC, *bc* are parallel, and so on how many sides soever there be.

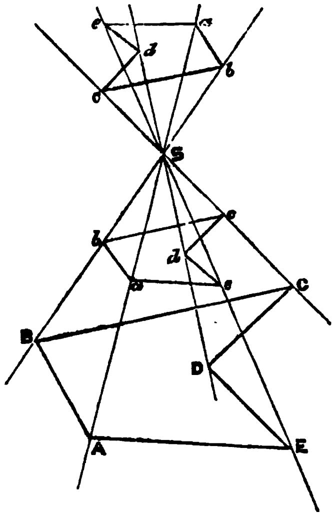
Whence again, since AB, BC which meet in B are parallel to *ab*, *bc* which meet in *b*, the angle *abc* is equal to ABC (*Prop.* VII.). In the same manner all the other angles are equal each to each in the two figures.

Again, since A*a*, B*b*, C*c*, etc., are all parallel (*Def.* 9.), the figures AB*ba*, BC*cb*, etc., are parallelograms, and hence $ab = AB$, $bc = BC$, etc. Consequently,

$$AB : BC :: ab : bc,$$

or the sides about the equal angles ABC, *abc* are proportional; and so on for all the others. The figures are therefore similar.

But similar figures which have their homologous sides also equal, are altogether equal.



(2.) Let the pyramid SABCDE (S being the vertex, and ABCDE the base), be cut by a plane *abcde* parallel to the base: then the figure *abcde* will be similar to the base ABCDE.

For, as in the preceding case, *ab*, *bc*, *cd*, *de*, *ea* are respectively parallel to AB, BC, CD, DE, EA; and the angles *abc*, *bcd*, etc., equal to the angles ABC, BCD, etc. The figure *abcde* is therefore equiangular with the base.

Again, the triangles *a*S*b*, ASB are similar, since *ab* is parallel to AB; and similarly with respect to the other pairs. Whence

$$ab : AB :: bS : BS :: bc : BC, \text{ or}$$
$$ab : bc :: AB : BC;$$

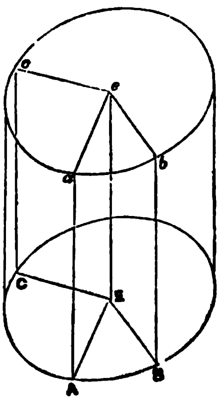
that is the sides about the equal angles *abc*, ABC are proportional.

In the same manner it may be proved for the sides about the other equal angles. The figure *abcde* is hence similar to ABCDE.

Scholium. The cutting plane may be *below* the base, between the base and vertex, or beyond the vertex. It is analogous to the three cases of *Euc.* VI. 2. In the last case, however, the figure will have a reversed position, as may be seen in the figure.

PROPOSITION XXIV.

If a cylinder or cone be cut by a plane parallel to the base, the section will be a circle; and in the cylinder this circle will be also equal to the base.

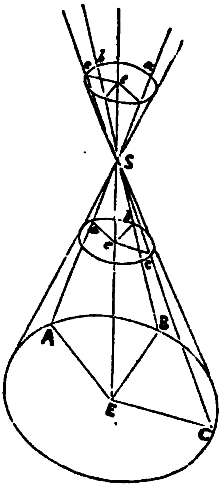


(1.) Let ABC be the base of a cylinder, and E its centre; let the cylinder be cut by a plane parallel to the base in *abc*: then *abc* is a circle.

For let E*e* be the intercepted portion of the axis, and A*a*, B*b*, C*c*, etc., the intercepted portions of the edges. All these, by the definition of the cylinder, are parallel; and being between parallel planes, they are equal (*Prop.* XV.); and each equal to E*e*.

Also, since the plane A*ae*E cuts the parallel planes in AE, *ae*, these lines are parallel (*Prop.* I.); and hence A*ae*E is a parallelogram, having *ae* = AE. In like manner in all the corresponding cases, *be* = BE, *ce* = CE, etc. But AE = BE = CE, etc.; and hence *ae* = *be* = *ce*, etc., and *e* is the centre of a circle, the radius of which is equal to that of the base. The section, therefore, is a circle equal to the base.

(2.) Let S be the vertex of a cone, of which the base is ABC and axis SE; and let it be cut by a plane in *abc;* then *abc* is a circle.



For draw the planes SEA, SEB, etc., cutting ABC in AE, BE, etc., and *abc* in *ae*, *be*, etc.

Then the parallel planes ABC, *abc*, being cut by SEA, the lines AE, *ae* are parallel (*Prop.* I.). In the same manner BE, *be* are parallel, and so on.

Wherefore the triangle S*ae* is similar to SAE, S*be* to SBE, etc.; and hence

$$ae : \mathrm{AE} :: e\mathrm{S} : \mathrm{ES} :: be : \mathrm{BE}, \text{ or}$$
$$ae : eb :: \mathrm{AE} : \mathrm{EB}.$$

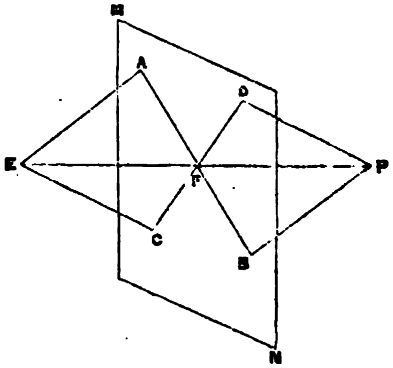
But E is the centre of the circle ABC, and hence AE = EB; and therefore *ae* = *eb*. In the same manner *ae* = *ec*, etc.

Consequently, the points *a*, *b*, *c*, etc., being in one plane, and all equidistant from a point *e* in it, they are in the circumference of a circle, of which *e* is the centre.

SCHOLIUM. The same remark applies to the different positions of the cutting plane in respect of the cone that was made respecting the pyramid.

PROPOSITION XXV.

If from two fixed points E, P any parallel straight lines be drawn, viz., EA and PB, EC and PD, etc., to meet the fixed plane MN in A and B, C and D, etc.: then AB, CD, etc., will always pass through the same point in the plane, viz., the point F in which EP meets it.



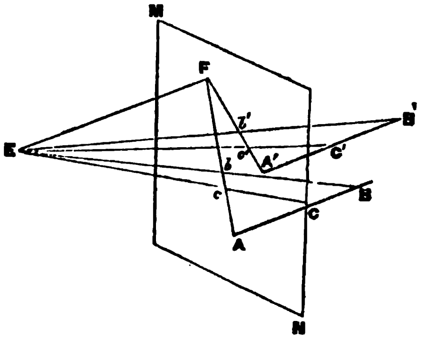
For since EA, PB are parallel, they are in one plane. Also the points E, P being in that plane, the line EP is in that plane. The point F being in the line EP is also in that plane; and hence the intersection AB of the plane EABP with MN contains the point F. That is, the line AB passes through F.

In the same manner all the other lines drawn as prescribed (as CD), pass through F.

Scholium. This is the foundation of *Dr. Brook Taylor's method of putting a point in perspective.*

PROPOSITION XXVI.

Let there be a point E on one side of the plane MN, and any number of parallel lines AB, A'B', etc., on the other; and let EF be drawn parallel to them meeting the plane MN in F: then lines drawn from points B, C, B', C', etc., in the lines AB, A'B' to the point E, all cut the plane MN in points b, c, b', c' in the lines AF, A'F; and lines drawn to every point in AB, A'B' etc., cut MN in points lying between A and F, A' and F, etc.

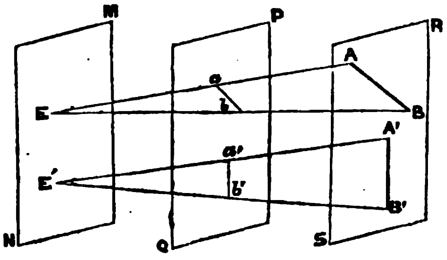


For AB, EF being parallel, they are in one plane; and hence EC, EB, etc., are in that plane. Whence they cut the plane MN. Their intersections with the plane MN are therefore in the line AF, in which ABEF intersects MN. In the same manner for the points B', C' in the line A'B'.

Scholium. This is expressed *in perspective* by saying that parallel lines AB, A'B', etc., "have the same vanishing point."

PROPOSITION XXVII.

If there be three parallel planes MN, PQ, RS, and any equal lines AB, A'B' be taken in one of them, RS, and from any points E, E' in MN lines EA, EB, E'A', E'B' be drawn to meet PQ in ab, a'b': then ab, a'b' will be equal.



For since the four straight lines EA, EB, E′A′, E′B′ are cut by the three parallel planes MN, PQ, RS, they are divided proportionally (Chap. I., 22). That is,

$$\mathrm{EA : EB :: E'A' : E'B' :: E}a : \mathrm{E}b :: \mathrm{E'}a' : \mathrm{E'}b'.$$

Also, since the planes AEB, A′E′B′ are cut by the parallel planes, the lines AB, A′B′ are respectively parallel to ab, $a'b'$; and the triangles AEB, A′E′B′ are respectively similar to $a\mathrm{E}b$, $a'\mathrm{E'}b'$.

$$\begin{aligned} \text{Wherefore } \mathrm{AE : AB} &:: a\mathrm{E} : ab, \\ \mathrm{A'E' : A'B'} &:: a'\mathrm{E'} : a'b'. \\ \text{But } \mathrm{AE} : a\mathrm{E} &:: \mathrm{A'E'} : a'\mathrm{E'}, \text{ and hence} \\ ab : a'b' &:: \mathrm{AB : A'B'}. \end{aligned}$$

But by hypothesis $\mathrm{AB = A'B'}$, and hence $ab = a'b'$.

Of course, if E and E′ be made to coincide, the same equality of ab to $a'b'$ holds good.

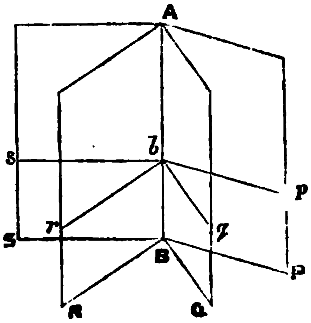
This proposition is the fundamental one of *Gravesandes's method of perspective.* This method, and that of Dr. Brook Taylor, have been advantageously combined by the late Peter Nicholson in his work on the subject.

CHAPTER II.

PERPENDICULARITY.

PROPOSITION I.

From any point in a given straight line innumerable perpendiculars may be drawn to that line.



Let B be a given point in the given line AB; innumerable perpendiculars to AB may be drawn from B.

For through AB innumerable planes may be drawn, as AP, AQ, AR, AS, etc. In *each* of these, one perpendicular can be drawn, as BP, BQ, BR, BS, etc. Whence the conclusion follows with respect to B.

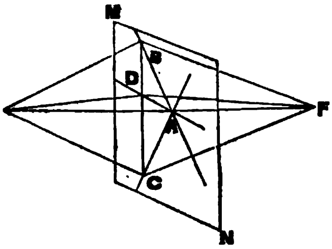
In the same manner from any other point b, the innumerable perpendiculars bp, bq, br, bs. etc. may be drawn in the same planes. The conclusion now follows universally.

PROPOSITION II.

If any number of lines be drawn in a plane through the same point, and if at that point another straight line be drawn perpendicular to two of them, it will be perpendicular to all the others.

In the plane MN let any number of lines AB, AC, AD, etc., pass through the point A; and let the line EF be perpendicular to any two of them AB, AC at A; then it will be perpendicular to AD, etc.

For, draw any line in the plane to cut AB, AC, AD in B, C, D. Make EA, AF equal; and join EB, EC, ED, and FB, FC, FD.



Then because EA, AB are equal to FA, AB and the angles EAB, FAB equal (being right angles), EB is equal to FB (*Euc.* I. 4).

Similarly, EC is equal to FC.

Wherefore the two sides EB, BC are equal to the two FB, BC, and the base EC to the base FC; and therefore the angle EBC to the angle FBC (*Euc.* I. 8).

Again, since EB, BD are equal to FB, BD and the angle EBD to the angle FBD, the bases ED, DF are also equal (*Euc.* I. 4).

And, lastly, since EA, AD are equal to FA, AD, and the base ED to the base FD, the angle EAD is equal to the angle FAD; and they are adjacent angles, and hence right angles; that is, the line EAF is perpendicular to AD.

In the same manner it is proved to be perpendicular to any, and therefore to every other line drawn in the plane MN through the point A.

Scholium. It is on account of this theorem that a line perpendicular to a plane is often *defined* as "that which is perpendicular to any line in the plane drawn through the point of intersection."

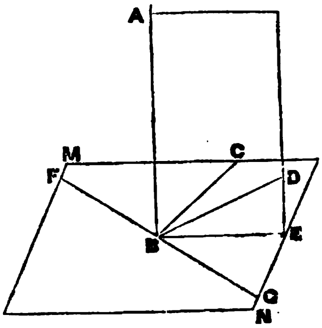
As a *definition*, this is open to the same objection that has been often made against some of Euclid's other definitions—that *it involves superfluous conditions.* Perpendicularity to two of the lines completely defines the perpendicular; perpendicularity to all is a demonstrated property, and should be quoted as such.

When, however, a line is predicated to be perpendicular to a plane, or a plane to a line, we are at full liberty to quote this property, without naming the specific lines through which the plane is drawn.

PROPOSITION III.

All lines perpendicular to the same line at the same point in it are situated in the same plane.

Let any number of lines BC, BD, BF, etc. be perpendicular to AB at the point B; they will all lie in the same plane.



For, through any two of them BC, BF let a plane MN be drawn; and if any one of the perpendiculars (as BD) be not in that plane, it will be either above or below it. Let it be above, as in the diagram; and through AB and BD draw the plane AE cutting MN in BE.

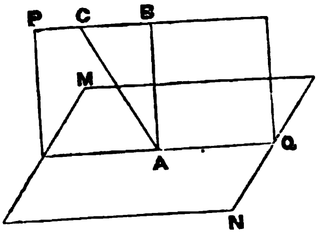
Then because AB is perpendicular to BC, BF, it is perpendicular to BE (*Prop.* II.), and by hypothesis ABD is a right angle: whence the angles ABD, ABE in the same plane AE are equal to one another, the less to the greater, which is impossible.

Whence BD does not lie above the plane MN; and in the same way it does not lie below it. It is therefore in the plane MN.

The same demonstration applies to every other line perpendicular to AB at B; and hence the proposition is true.

PROPOSITION IV.

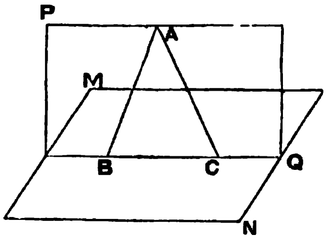
From the same point there can be drawn only one perpendicular to a plane, whether that point be in the plane or without it.



(1.) Let A be a point *in* the plane MN: there can only be drawn from A one perpendicular to the plane MN.

For, if possible, let there be two, viz., AB, AC, and through them draw the plane PQ, cutting MN in AQ.

Then since BA is perpendicular to the plane MN, BAQ is a right angle (*Prop.* II.); and since CA is perpendicular to MN, CAQ is a right angle. The angles BAQ, CAQ in the same plane PQ are therefore equal; the less to the greater, which is impossible. Whence from A situated in the plane MN only one perpendicular can be drawn.



(2.) Let A be a point *without* the plane MN, only one line can be drawn from A perpendicular to MN.

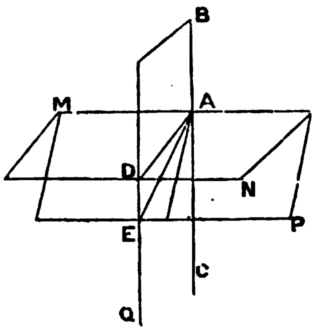
For, if possible, let there be two, viz., AB, AC, and through them draw a plane PQ, cutting MN in BCQ.

Then since AB, AC are perpendicular to the plane MN, the angles ABC, ACB of the triangle ABC are two right angles (*Prop.* II.); which is impossible (*Euc.* I. 17). Whence both lines cannot be perpendicular to the plane MN.

Whence, whether the point be in the plane or without it, only one line through the point can be perpendicular to the plane.

PROPOSITION V.

Through the same point only one plane can be drawn perpendicular to a line, whether that point be in the line or without it.

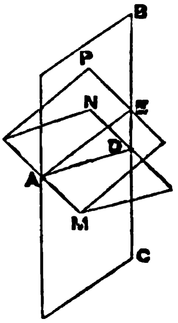


(1.) Let A be a point *in* the line BC: only one plane perpendicular to BC can be drawn through A.

For, if possible, let there be two planes MN, MP drawn through A perpendicular to BC; and let AM be their intersection; and draw any other plane BQ through BC, and not passing through AM, cutting MN and MP in AD and AE.

Then since BC is perpendicular to the planes MN, MP, the angles BAD, BAE are right angles (*Prop.* II.); and they are in one plane BQ, which is impossible. Whence MN and MP cannot both be perpendicular to BC.

(2.) Let A be *without* the line BC; only one plane can be drawn through A perpendicular to BC.

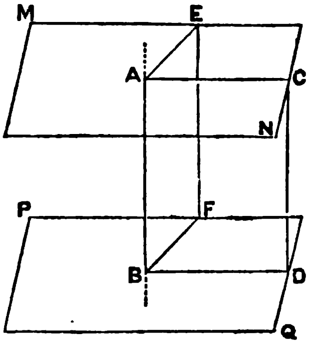


For, if possible, let there be two, MN, MP, the common section of which is AM; and let a plane be drawn through BC and A, cutting MN, MP in AD, AE. Then since EA is in the plane MP perpendicular to CB, the angle AEB is a right angle; and since DA is in the plane MN perpendicular to CB, the angle BDA is a right angle. But AD, AE, and BC are in the same plane. Wherefore the interior angle ADE of the triangle ADE is equal to the exterior; which is impossible (*Euc.* I. 16.). Wherefore only one plane through A can be perpendicular to BC.

PROPOSITION VI.

(1.) *If two planes be perpendicular to a line, they are parallel to one another; and*

(2.) *If one of two parallel planes be perpendicular to a line, the other is also perpendicular to it.*



(1.) Let the planes MN, PQ be perpendicular to AB: they are parallel to one another.

For, through AB draw any two planes AD, AF, cutting MN in AC and AE, and PQ in BD and BF.

Then since AB is perpendicular to MN, the angles BAC, BAE are right angles (*Prop.* II.); and since AB is perpendicular to PQ, the angles ABD, ABF are right angles.

Then, since in the same plane AD, the lines AC, BD make right angles with AB, they are parallel. Similarly AE, BF are parallel. Wherefore the two lines CA, AE which meet in A being parallel to the two DB, BF which meet in B, the planes MN, PQ which contain them are parallel (*Prop.* VII. *Chap.* I.).

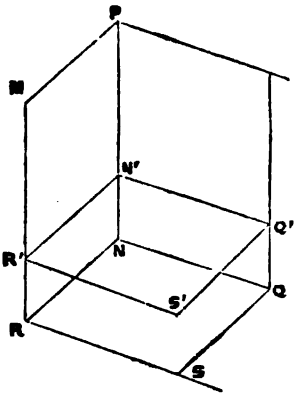
(2.) Let MN, PQ be parallel planes, and one of them MN be perpendicular to the line AB: then the other, PQ, will also be perpendicular to AB.

For draw any two planes AD, AF through AB cutting the planes in AC, AE and BD, BF as before.

Then since the planes MN, PQ are parallel, the line AC is parallel to BD and AE to BF. Whence the two angles EAB, ABF are together equal to two right angles; and one of them EAB is a right angle (*Prop.* II.); and consequently the other, ABF is a right angle, or BF is perpendicular to AB. In the same manner BD is perpendicular to AB; and hence (*Def.* 5) the plane PQ through BD, BF is perpendicular to AB.

PROPOSITION VII.

The traces of the profile angle are perpendicular to the edge of the dihedral angle.



Before proceding further, let the student refer to *Defs.* 3, 4, 7, and (*Prop.* II. *Chap.* II.); he will then see that this proposition is only the proper technical form of the following:—

A plane perpendicular to the edge of a dihedral angle cuts the faces in lines which are perpendicular to that edge.

Let MN, PQ be the faces of the dihedral angle MNQP, PN its edge, and RNQS a profile plane whose traces are RN, NQ: then RN, NQ will be perpendicular to PN.

For, since PN is perpendicular to the plane RQ, it is perpendicular to every line in it drawn through N: that is RN, NQ are both perpendicular to PN.

PROPOSITION VIII.

(1.) *The profile traces at all points in the edge of a dihedral angle are parallel, those in one face to one another, and those in the other to one another; and*

(2.) *All the profile angles of the same dihedral angle are equal to one another.*

(Preceding figure.)

(1.) Let RQ, R'Q' be two profile planes to the dihedral angle MNQP; then the trace R'N' is parallel to the trace RN, and the trace N'Q' to the trace NQ.

For RN, R'N' being in the same plane, and both perpendicular to PN, they are parallel to one another.

In the same manner N'Q' is parallel to NQ; and that which is proved for *any one* point N' is true for *all other* points.

(2.) The profile angle R'N'Q' is equal to the profile angle RNQ.

For, since the two lines R'N', N'Q' which meet in N' are parallel to the two RN, NQ which meet in N, the angle R'N'Q' contained between the first two is equal to the angle RNQ contained between the other two (*Prop.* VII. *Chap.* I.).

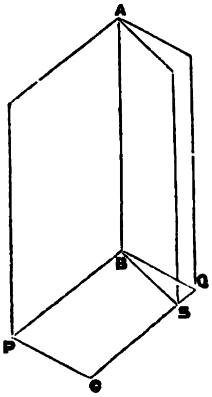
COR. All the profile planes of the same dihedral angle are parallel, that is, the plane R'Q' is parallel to RQ (*Prop.* VII. *Chap.* I.).

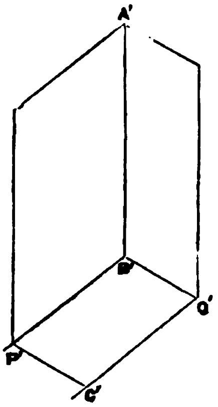
SCHOLIUM. It might have been inferred from this demonstration that all planes perpendicular to the same straight line are parallel; but this is usually made the subject of a separate proposition, and it is so done in this course.

PROPOSITION IX.

(1.) *If two dihedral angles be equal, their profile angles are equal; and*

(2.) *If the profile angles of two dihedral angles be equal, the dihedral angles are themselves equal.*





(1.) Let PABQ, P'A'B'Q' be two equal dihedral angles, and PQ, P'Q' the profile planes; PB, BQ the profile traces of the first dihedral angle, and P'B', B'Q' those of the second: then the angles PBQ, P'B'Q' will be equal.

For let the dihedral angle P'A'B'Q' be transposed to the position such that A'B' shall coincide with AB, the point B' coinciding with B, and the plane A'P' with the plane AP.

Then, since the face A'P' coincides with AP, and the dihedral angles are equal (*Hypoth.*), the face A'Q' will coincide with the face AQ.

Again, since A'B' coincides with AB, the point B' with the point B, and the face A'P' with the face AP, the perpendiculars B'P', BP in those coincident faces must coincide. In like manner, B'Q' must coincide with BQ; and hence the angles P'B'Q', PBQ will be equal.

(2.) If the profile angles P'B'Q', PBQ be equal, the dihedral angles P'A'B'Q', PABQ will be equal.

For so transpose the dihedral angle P'A'B'Q' that the edge A'B' shall coincide with AB, the point B' with B, and the plane A'B'P' with APB: then if the plane A'Q' do not coincide with AQ, let it take some other position as AS, and B'Q' the position BS.

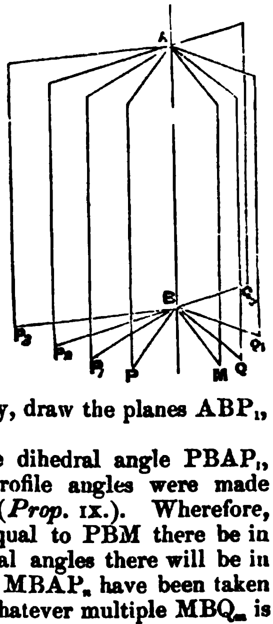
Then, as in the preceding case, B'P' coincides with BP; and since A'B'Q' is a right angle, ABS is a right angle, and BS is in the plane PQ (*Prop.* III.).

Whence since the dihedral angles PABS, P'A'B'Q' are equal, the profile angles P'B'Q', PBS are equal (*preceding case*); and P'B'Q' is equal to PBQ by hypothesis. Whence PBS is equal to PBQ, the less to the greater, which is impossible. Whence the dihedral angles P'A'B'Q', PABQ are not unequal; that is, they are equal.

Scholium. The properties in this and the preceding proposition are *tacitly assumed* by Euclid, and by most other writers; but very little reflection is necessary to convince us that such a course is altogether unwarranted under the circumstances of the case. It should be laid down as a general rule in mathematics, that *propositions which admit of proof from principles already laid down, ought never to be assumed as independently true.* It is tantamount to multiplying our axioms without necessity; and moreover with great risk of involving contradictions amongst them.

PROPOSITION X.

Two dihedral angles have the same ratio to one another that their profile angles have to one another.



Let PABM, MBAQ be two dihedral angles, whose profile angles are PBM, MBQ: then

PABM : MABQ : : PBM : MBQ.

[In the figure, the dihedral angles are placed contiguously, so as to have one face common to both of them, viz., BM. When this is not so given, we must conceive them to be so placed by transposing one of them nearly as in the preceding proposition.]

For since PB, MB, QB are perpendicular to AB (*Def.* 4) they are in one plane (*Prop.* III.). In that plane take any number of angles, PBP_1, P_1BP_2 etc., each equal to PBM; and any number QBQ_1, Q_1BQ_2, etc., each equal to MBQ; and, finally, draw the planes ABP_1, ABP_2, etc., and ABQ_1, ABQ_2, etc.

Then PBP_1 is the profile angle of the dihedral angle $PBAP_1$, P_1BP_2, of P_1BAP_2, etc.; and as the profile angles were made equal, the dihedral angles are also equal (*Prop.* IX.). Wherefore, whatever number of profile angles each equal to PBM there be in MBP_n,* the same number of equal dihedral angles there will be in $MBAP_n$; that is, equimultiples MBP_n and $MBAP_n$ have been taken of MBP and MBAP. In like manner, whatever multiple MBQ_m is of MBQ, the same multiple will $MBAQ_m$ be of MBAQ.

Again, if the profile angle MBP_n be greater than MBQ_m the dihedral angle $MBAP_n$ will be greater than $MBAQ_m$; if equal, equal; and if less, less.

Wherefore (*Euc.* V. *Def.* 5.)

PBAM : QBAM : : PBM : QBM.

SCHOLIUM. It is essential to the acquisition of facility in the application of these principles to give close attention to some special cases that arise.

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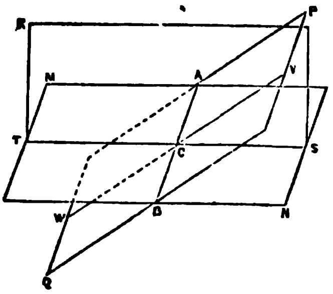
* This notation is used to signify that there are *n* dihedral angles with their corresponding profile angles taken, without specifying any particular number. The same with respect to *m* in the other set of multiples.



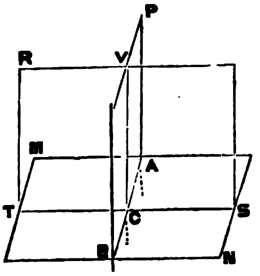
(1.) Let MN, PQ be two planes which cut one another in AB; then four dihedral angles will be formed, viz., NBAP and its opposite, MABQ, and the opposite pair, PABM and QBAN.

Let any profile plane RS be drawn to these two planes, cutting MN in TCS and PQ in VCW.

Then, since the opposite profile angles VCS, WCT are equal, the corresponding dihedral angles are equal (*Prop.* IX.), viz., PABN and QBAM. In the same manner the opposite dihedral angles PABM, NBAQ are equal. That is, if two planes cut each other, the opposite dihedral angles are equal.

(2.) If two planes, QA, BP on opposite sides of another plane MN meet in the same line AB and make the opposite angles (either the dihedral or the profile) equal, those planes are one and the same plane.

(3.) If a plane BP meet a plane MN in AB, the dihedral angles PABN, PABM have the same ratio as the profile angles VCS, VCT.



(4.) When the adjacent dihedral angles are equal, each of them is a dihedral right angle, by *Def.* 4; and in this case each of the adjacent profile angles is a plane right angle. And conversely when the adjacent profile angles are equal, each is a right angle; and the corresponding dihedral angles are equal, and each a right angle. The one condition leads to the other; and either being given by hypothesis, the other may be inferred in the reasonings towards any specific conclusion.

(5.) The sum of all the dihedral angles that can be formed about a line are together equal to four dihedral right angles.

(6.) It is on account of the proportionality between two dihedral angles and their profile angles that the latter have been used instead of the former in all the reasonings of the ancients, and in those of most of the moderns too. The dihedral angle has, indeed, been scarcely recognised as a subject of geometry at all; and the profile angle has been almost universally substituted for it, under the name of the *inclination* of one plane to the other. This has been productive of much confusion in the student's mind, and created great difficulty in his appreciating the force of the argument in cases where these angles occur. Even into a simple case of the dihedral right angle, or a plane perpendicular to a plane, it has introduced a degree of confusion that renders the subject difficult to comprehend, and indeed puts the whole inquiry in a shape so vague that very few persons are ever able to acquire a clear and comprehensive view of the subject.

PROPOSITION XI.

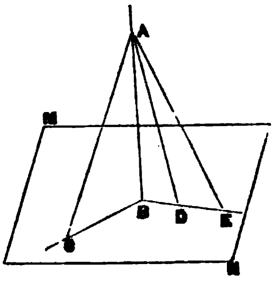
If from a point without a plane, straight lines be drawn to meet the plane: then

(1.) *The perpendicular is the shortest line that can be drawn;*

(2.) *Those which meet the plane at equal distances from the perpendicular are equal;*

(3.) *That which meets it at the less distance is less than the one that meets the plane more remotely.*

Let A be a given point without the given plane MN; AB perpendicular to MN; AC, AD, lines meeting the plane at equal distances BC, BD from B; and AE meeting MN at a distance BE from B greater than BC: then,



(1.) AB will be less than any one AC of the other lines.

For draw BC:—

Then ABC is a right angle; and consequently greater than the angle ACB of the triangle ABC. Wherefore AC opposite the greater angle is greater than AB opposite the less (*Euc.* I. 19.). That is, AB is less than AC.

(2.) The lines AC, AD are equal.

For, since AB is perpendicular to the plane MN, the angles ABC, ABD are right angles. Therefore the two sides AB, BC are equal to the two AB, BD, and the included angles are equal; and hence (*Euc.* I. 4.) the bases AC, AD are equal.

(3.) Since BE is greater than BC, cut off a part BD equal to BC, and join AD.

Then, since ABD is a right angle, ADE is greater than a right angle (*Euc.* I. 17); and hence AED is less than a right angle. Wherefore ADE is greater than AED; and hence AE greater than AD. But AD is equal to AC by the preceding case; and hence AE is likewise greater than AC, or AC less than AE.

Scholium. Several corollaries, or rather easy deductions, follow from this simple property. They are put down here as *exercises* for the student; and they should be established here, as some of them come into use hereafter, and will be quoted by reference to this place.

Cor. 1. Equal lines AC, AD make equal angles with the perpendicular AB; and likewise equal angles with the lines BC, BD drawn to B.

Cor. 2. All equal lines drawn from a point A without the plane MN meet the plane in the circumference of a circle described about B as centre.

Cor. 3. The greater line AE makes a greater angle BAE with the perpendicular AB than the less AC does.

Cor. 4. If lines be drawn from points C, D equidistant from B to the same point A in the perpendicular AB, they will be equal; and they will make equal angles with the perpendicular, and equal angles with the lines drawn to B.

Some others might be added; but it may be stated here generally, that any one condition being given, the corresponding one follows, either by direct or *ex absurdo* deduction, at once.

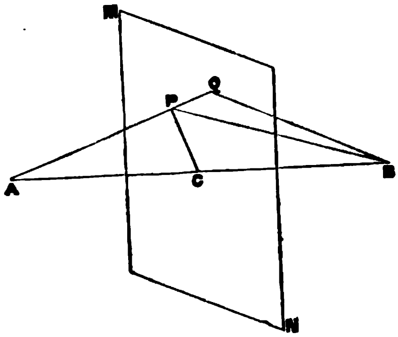
PROPOSITION XII.

A plane is drawn perpendicular to a given line at its middle point: then

(1.) *Any point in the plane is equidistant from the extremities of the line; and*

(2.) *All points equidistant from the extremities of the line are situated in that plane.*

(1.) Let AB be a straight line, C its middle point, and MN a plane drawn through C perpendicular to AB: then any point P in MN will be equidistant from A and B.



For join PA, PB, PC.

Then since AB is perpendicular to the plane MN, meeting it in C, and CP is in the same plane, the angles ACP, BCP, are right angles, and therefore equal to one another (*Prop.* II.). Wherefore the sides AC, CP are equal to BC, CP and the included angles equal; and (*Euc.* I. 4) hence AP is equal to PB.

(2.) Let Q be any point equidistant from A and B: it will lie in the plane MN.

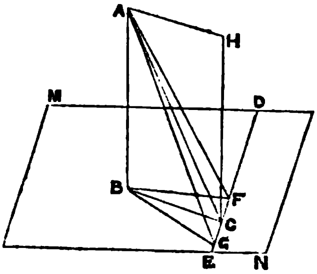
For join QA, QB; and, if possible, let Q be not in the plane MN, but on one side of it—suppose the same side with B. Then QA will cut MN in some point, P. Join PB.

Then, by the preceding case, PA is equal to PB; and hence adding PQ to both, AQ is equal to BP and PQ together. But BP and PQ together are greater than BQ (*Euc.* I. 20); and hence AQ is greater than BQ, which is absurd.

PROPOSITION XIII.

If a line be perpendicular to a plane, and from its intersection with the plane a line be drawn perpendicular to any line in the plane; then the plane through the perpendiculars will be perpendicular to the line in the plane.

Let AB be perpendicular to the plane MN, and let BC be drawn



perpendicular to the line DE in that plane: then the plane through AB, BC will be perpendicular to DE.

For in DE take on opposite sides of C the equal segments CF, CG; and draw FB, GB to B, and FA, GA to any other point A in the perpendicular.

Then since BC, CF are equal to BC, CG each to each, and the included angles BCF, BCG equal (being right angles by hypothesis) BF is equal to BG (*Euc.* I. 4.).

Again, since F and G are equidistant from B, the lines AF, AG are equal (*Prop.* XI.). Wherefore the sides AC, CF are equal to the sides AC, CG, and the base AF to the base AG; and hence the angle ACF is equal to its adjacent angle ACG. Whence AC is perpendicular to DE, but BC is also perpendicular to DE by hypothesis; and hence DE is perpendicular to both BC and CA; and hence to the plane BH drawn through them (*Prop.* II.). And this plane is that drawn through the perpendiculars AB, BC; since AC is in the same plane with them.

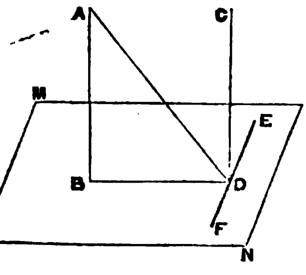
Cor. If the same hypothesis be made, all lines drawn from C to meet AB are perpendicular to DE.

This is proved above, and is often given as the principle theorem; whilst that which has been enunciated is sometimes made a corollary from it.

PROPOSITION XIV.

(1.) *If two straight lines be perpendicular to a plane, they are parallel to one another; and*

(2.) *If two lines be parallel, the plane which is perpendicular to one of them will be perpendicular to the other.*



(1.) Let AB, CD be perpendicular to the plane MN: they will be parallel to one another.

For join BD; draw a line from D to any point A in AB; and in the plane MN the line EDF perpendicular to BD.

Then since AB is perpendicular to MN, and BD to the line EF in MN, the plane ABD is perpendicular to EF (*Prop.* XIII.); that is DB and DA are each perpendicular to EF. But CD is also perpendicular to the plane MN, and hence CD is perpendicular EF; that is, EF is perpendicular to DB, DA, DC at the point D. Wherefore the three lines DC, DA, DB being perpendicular to EF at the point D, they are in one plane (*Prop.* III.). But AB is in the same plane with DA, DB, and therefore with CD; or the two lines perpendicular to the plane MN are themselves in one plane.

Again, since AB, CD are perpendicular to the plane MN, the angles ABD, CDB are right angles. Whence these lines being in the same plane ABDC, and perpendicular to the same line BD in it, they are parallel.

(2.) Let AB, CD be parallel, and the plane MN be perpendicular to one of them AB, it will be perpendicular to the other CD.

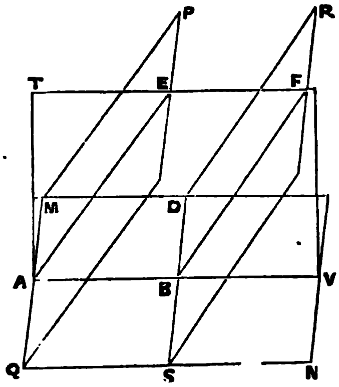
Since AB, CD are parallel, they are in one plane. Let this intersect MN in BD, and draw EF through D perpendicular to BD.

Then, since AB is parallel to CD, and ABD a right angle (AB being perpendicular to MN) the angle BDC is also a right angle, or CD is perpendicular to the line BD in the plane MN.

Again, since AB is perpendicular to MN and BD to a line EF in MN; the line EF is perpendicular to the plane ABD (*Prop.* XIII.); that is to the plane ABDC. Whence CD is perpendicular to EF; and it has been proved to be perpendicular to DB. It is hence perpendicular to the plane MN which contains the lines BD, EF (*Prop.* II.).

PROPOSITION XV.

If a plane cut two parallel planes, it makes equal dihedral angles with them.



Let the plane MN cut the parallel planes PQ, RS in MQ, DS; the dihedral angles PMQN, RDSN will be equal.

For since the planes PQ, RS are parallel, the edges MQ, DS of the dihedral angles are parallel (*Prop.* I. *Chap.* I.). Draw then a profile plane TV to one of the angles through A, to cut the system of planes in ABV, AE, BF.

Then since PQ, RS are parallel, and TV is perpendicular to one of them PQ, it is perpendicular to the other RS. It is hence also a profile plane of both the dihedral angles.

Again, since PQ, RS are parallel planes cut by the profile plane TV, the lines AE, BF are parallel (*Prop.* I. *Chap.* I.), and AV is common to both the profile angles. Wherefore the angle EAV is equal to the angle FBV; and hence the profile angles are equal. Wherefore, finally, the dihedral angles are equal.

COR. 1. If the parallel planes be produced, the alternate dihedral angles will be equal; and the two interior dihedral angles are together equal to two dihedral right angles. For this is the case with their profile angles (*Euc.* I. 29) and hence with the dihedral angles themselves.

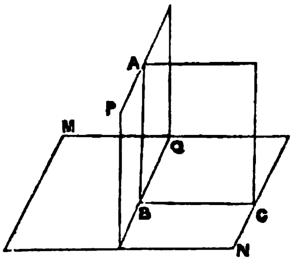
COR. 2. The converse, when enunciated in an unrestricted manner, is not true. But put thus, it is true:—

If two planes meet a third plane *in parallel lines*, and make with it equal dihedral angles *in the same direction*, those planes will be parallel. The italics indicate the restrictions alluded to.

COR. 3. The opposite dihedral angles of a parallelopiped are equal. (*Prop.* XX. *Chap.* I.).

PROPOSITION XVI.

If a line be perpendicular to a plane, every plane drawn through it will be perpendicular to that plane.



Let the line AB be perpendicular to the plane MN: then any plane PQ drawn through AB will be perpendicular to MN.

For let the plane PQ cut MN in BQ, and in MN draw the line BC perpendicular to BQ; and also draw the plane AC through AB, BC.

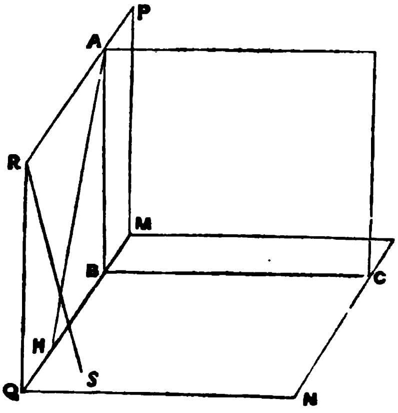
Then since AB is perpendicular to MN, it is perpendicular to BQ; and BC is perpendicular to BQ by construction. Whence ABC is the profile angle of the dihedral angle PQBN (*Def.* 4).

But AB being perpendicular to MN, ABC is a right angle (*Prop.* II.); and the profile angle of the planes being a right angle, then the dihedral angle is a right angle (*Def.* 4). That is, the plane PQ is perpendicular to MN.

PROPOSITION XVII.

If two planes be perpendicular to one another, every line drawn in one plane perpendicular to the common section will be perpendicular to the other plane; and every line drawn from a point in one plane perpendicular to the other plane lies wholly in the former, and is perpendicular to the common section.

(1.) Let the plane PQ be perpendicular to the plane MN; and let the line AB be drawn in one plane PQ perpendicular to MQ the common section of the planes: then AB will be perpendicular to the other plane MN.



For in the plane MN, draw BC perpendicular to the common section MQ. Then since AB, BC are perpendicular to MQ, the plane through them is the profile plane, and ABC the profile angle. But the dihedral angle PQMN is a right angle (*Hyp.*), and hence ABC is a right angle (*Prop.* x. *Scholium.*).

But AB is perpendicular to MQ (*Hyp.*), and hence it is perpendicular to the two lines BC, BM in the plane MN: that is, to the plane MN itself.

(2.) All lines drawn from points, as R, in the plane PQ perpendicular to the plane MN, will be wholly in the plane PQ.

For, if possible, let some one of them RS be without the plane PQ. From R draw RQ perpendicular to the line QM.

Then by the preceding case RQ is perpendicular to MN. But by admission RS is also perpendicular to MN: that is, two lines RQ, RS can be drawn from the same point R perpendicular to the plane MN. This is impossible (*Prop.* iv.); and hence the perpendicular to MN falls wholly in the plane PQ.

(3.) All perpendiculars from points in PQ to the plane MN are perpendicular to the line MQ.

For if possible let AH be drawn in the plane PQ perpendicular to MN, yet not perpendicular to MQ; and draw AB perpendicular to MQ.

Then reasoning as in the last case, the conclusion follows.

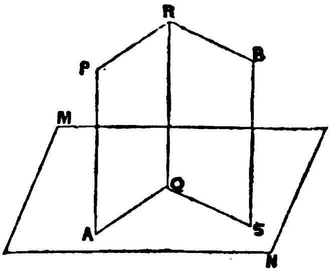
PROPOSITION XVIII.

(1.) *If two planes which intersect be perpendicular to a third plane, their intersection will also be perpendicular to this third plane; and*
(2.) *If a plane be perpendicular to the intersection of two other planes, it will be perpendicular to each of those planes.*

(1.) Let the two planes PQ, RS which intersect in RQ, be both perpendicular to the plane MN; then will RQ be perpendicular to the plane MN.

For, from any point A in the intersection AQ, draw a line AP in PQ perpendicular to AQ; and similarly a line SB in the plane RS perpendicular to QS.

Then (*Prop.* xvii.), AP, SB are perpendicular to MN; and hence (*Prop.* xiv.) they are parallel to one another.



Also, since the planes PQ, RS which intersect in RQ are drawn through the parallel lines AP, BS, the intersection QR is parallel to both of them (*Prop.* ii. *Chap.* i.).

And, finally, since AP is perpendicular to MN, and QR parallel to AP, it follows that QR is also perpendicular to MN.

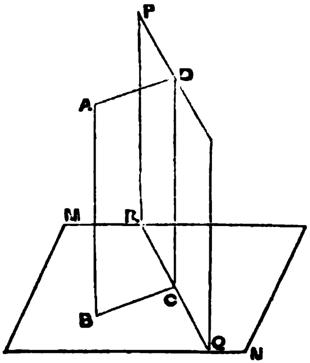
(2.) Let the plane MN be perpendicular to QR the intersection of the two planes PQ, RS: then it will be perpendicular to the planes PQ, RS themselves.

For, since RQ is perpendicular to MN, the planes PQ, RS passing through it are also perpendicular to MN (*Prop.* xvi.): that is, MN is perpendicular to PQ and RS.

PROPOSITION XIX.

(1.) *If a line and plane be both perpendicular to the same plane, they will be parallel to one another; and*

(2.) *If a line and plane be parallel, every plane perpendicular to the line will be also perpendicular to the plane.*



(1.) Let the line AB and plane PQ be both perpendicular to the plane MN: then they will be parallel to one another.

For from any point C in the intersection QR of the planes, draw CD in the plane PQ perpendicular to RQ.

Then since the plane PQ is perpencular to MN, the line CD is also perpendicular to MN (*Prop.* xvii.); and therefore parallel to AB (*Prop.* xiv.).

Then since AB is parallel to the line CD in the plane PQ, it is parallel to the plane PQ itself (*Prop.* iii. *Chap.* i.).

(2.) Let AB be parallel to the plane PQ, and perpendicular to the plane MN: then MN will be perpendicular to PQ.

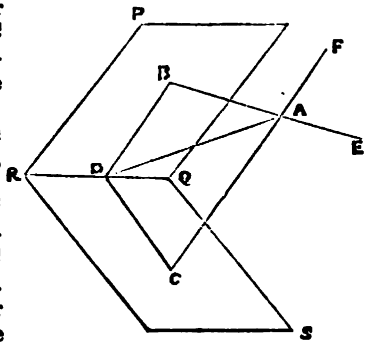
For draw any plane through AB to cut PQ in CD.

Then since AB is parallel to the plane PQ, the line CD is parallel to AB (*Prop.* ii. *Chap.* i.); and since AB is perpendicular to MN, and CD parallel to AB, CD is also perpendicular to MN (*Prop.* xiv.).

Wherefore, again, the plane through CD is perpendicular to the plane MN (*Prop.* xvi.).

PROPOSITION XX.

If two planes intersect, and from any point lines be drawn perpendicular to them, then these lines will contain an angle equal or supplementary to the profile angle of the planes.



Let the planes PQ, RS intersect in RQ; and from any point A let lines AB, AC be perpendicular to PQ, RS: then the angle BAC will be either equal or supplementary to the profile angle of the planes.

For draw the plane through AB, AC, cutting PQ, RS in BD, CD.

Then since the plane ABDC passes through AB, and AB is perpendicular to PQ, the plane ABDC is perpendicular to PQ (*Prop.* xvi.). In like manner it is perpendicular to RS; and hence also to the intersection RQ of PQ, RS (*Prop.* xviii.). It is, therefore, the profile plane of the dihedral angle PQRS; and BDC is the profile angle.

Now the four angles of the quadrilateral figure ABDC are together equal to four right angles; and two of them ABD, ACD are right angles (*Prop.* II.): whence the other two BDC and BAC are together equal to two right angles; that is, BAC is the *supplement* of BDC.

If, however, we produce either of the lines AB or AC to E or F, the adjacent angle CAE or BAF is *equal* to the profile angle BDC. It is generally understood in this proposition, however, that the angle BAC turned towards the dihedral edge, is that specified.

But when the point A is not between the planes PQ, RS as in the preceding figure; but without the dihedral angle, the profile angle BDC is *equal* to the angle BAC formed by the perpendiculars.

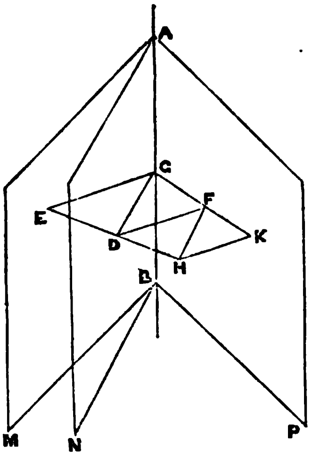
This will always be decided by joining AD: for when B, C, are on opposite sides of AD, the angle BAC is the supplement of BDC; and when on the same side of AD, BAC will be equal to BDC.

PROPOSITION XXI.

(1.) *If a dihedral angle be bisected, any point in the bisecting plane is equally distant from the faces of the angle; and*

(2.) *All points, each of which is equidistant from the faces of a dihedral angle are situated in the plane which bisects that angle.*

(1.) Let the plane AN bisect the dihedral angle MABP: then any point D in AN will be equidistant from AM and AP.



For draw the perpendiculars DE, DF to AM, AP, and through them the plane DEGF, cutting the faces in EG, GF, the edge in G, and the bisecting plane in DG.

Then, reasoning as in the preceding proposition, it is shown that DEGF is the profile plane of the given dihedral angle MABP.

Also, since the plane EGFD is perpendicular to AB, GD is also perpendicular to AB, and EGD, FGD are the profile angles of MABN, NABP. But these dihedral angles are equal (*Hypoth.*) and hence the profile angles EGD, DGF are equal. The angles DEG, DFG, also are right angles (*Prop.* II.); and hence the triangles are equiangular, and right angled; and have the common, hypothenuse GD. Wherefore the other sides are equal each to each, namely, those which are opposite to the equal angles; that is, the distance DE is equal to the distance DF.

(2.) Every point equidistant from the faces AM, AP, lies in the bisecting plane AN.

For if it be possible let any point H without the bisecting plane AN be equidistant from the faces AM, AP; and let HE, HK be those distances. Let HE cut AN in D, and draw DF perpendicular to AP, and join HF.

Then by the preceding case ED is equal to DF, and therefore EH (which is equal to ED and DH) is equal to HD and DF together. But HD and DF together are greater than HF; and hence EH is greater than HF.

Again, HKF is a right angle, and hence HFK is less than a right angle; and therefore HF is greater than HK. But HE has been proved to be greater than HF; therefore it is greater than HK. But it has been admitted to be equal to HK, which is contradictory. The point H, then, without the bisecting plane is not equidistant from the faces of the angle; and hence all equidistant points are in the bisecting plane.

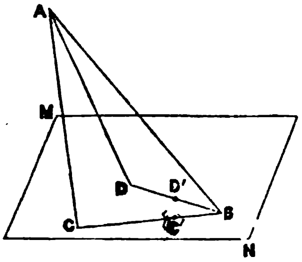
CHAPTER III.

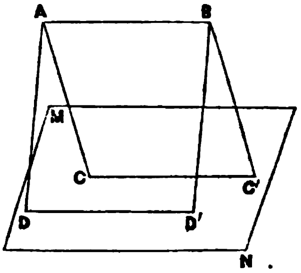
OBLIQUE LINE AND PLANE.

PROPOSITION I.

Except a line be perpendicular to a plane, only one plane can be drawn through it perpendicular to the plane.

The two other positions besides perpendicular which a line can have with respect to a plane, are oblique and parallel. If, then, it be possible that through such lines AB, more than one plane perpendicular to the plane MN can be drawn, let there be two; as BAC, BAD, the figures taken in order representing AB oblique to MN, and AB parallel to MN, respectively. And from any point A in AB, draw lines AC, AD perpendicular to CC′ and DD′ in the planes ACC′, ADD′.





Then, because ACC′ is perpendicular to the plane MN, and AC is drawn in ACC′ perpendicular to the common section of the planes CC′, it is perpendicular also to the plane MN (*Prop.* xvii.). In like manner AD is perpendicular to MN. Wherefore from the same point A, two straight lines have been drawn perpendicular to the plane MN; which (*Prop.* iv. *Chap.* ii.) is impossible.

The planes ACC′, ADD′ cannot therefore be both perpendicular to MN; that is, only one plane ABC can be drawn through AB perpendicular to MN.

Scholium 1. When AB is perpendicular to MN, it has been already shown (*Prop.* XVI.) that *all* planes through it are perpendicular to MN.

Scholium 2. The angle ABC which (*Def.* 6) measures the inclination of the oblique line AB to the plane MN, is called the acute *profile angle* in respect of this line; the perpendicular plane ABC, the *profile plane*, and BC the *profile trace*.

PROPOSITION II.

If a line be oblique to a plane, and lines be drawn in the plane through the point of intersection making angles with the oblique line: then

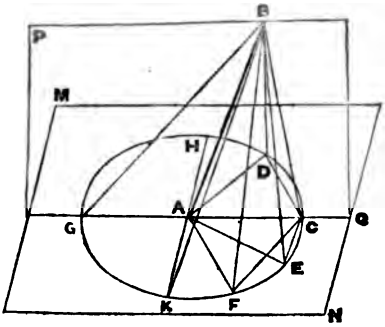
(1.) *Of all the angles which are so formed, the greatest and least are the profile angles of the line:*

(2.) *Only two lines can be drawn to make equal angles with the oblique line, one on each side of the profile plane:*

(3.) *Of these lines that makes the less angle with the oblique line, where the corresponding line in the plane makes the less angle with the acute profile trace, and conversely:*

(4.) *Only one line can be drawn in the plane perpendicular to the oblique line.*

Let the line AB be oblique to the plane MN; and let the profile plane PQ cut MN in GC; and let other lines AD, AE, AF, etc., be drawn in the plane MN: then,



(1.) The acute profile angle BAC is less than another angle BAE, and the obtuse profile angle BAG is greater.

For, about A in the plane MN describe any circle GHC, cutting the several lines drawn from A in that plane; and let BC be drawn in the profile plane ABQ perpendicular to AQ; and join AE, AG.

Then, since BC is perpendicular to the plane MN (*Prop.* I. *Schol.* 2), it is less than BE (*Prop.* XI. *Chap.* II.); wherefore the two BA, AC are equal to the two BA, AE each to each; but the base BC less than the base BE: wherefore (*Euc.* I. 25) the angle BAC is less than BAE. That is, BAC is the least angle that can be made under the circumstances.

Again, since CG is the diameter of the circle GHC, it is greater than any other line CE in it (*Euc.* III. 15); and hence BG is greater than BE (*Prop.* XI. *Chap.* II). Wherefore, BA, AG being equal to

BA, AE, but the base BG greater than BE, the angle BAG is greater than BAE (*Euc.* I. 25); that is, BAG is the greatest angle that can be made under the circumstances.

(2.) Only two lines AD, AE can be drawn in MN to make equal angles with AB.

For, let the lines AD, AE be drawn to make equal angles with GC, one on each side of it; and join AD, AE, CD, CE.

Then since the angles CAD, CAE at the centre of the circle are equal, the chords CD, CE which subtend them are equal (*Euc.* III. 29). Wherefore, CD, CE being equal, BD, BE are also equal (*Prop.* XI. *Chap.* II). Whence (*Euc.* I. 8) the angle BAD is equal to BAE; and two equal angles can be made.

It remains to show that a third angle cannot be made equal to either of these. For if possible, let BAF be equal to BAD or BAE; and join BF, CF.

Then, since CD, CE are equal, the line CF is greater or less than either of them. Suppose it greater, then BF is greater than BD or BE. Then BA, AF are equal to BA, AD, but BF greater than BD; and hence (*Euc.* I. 25) the angle BAF is greater than BAD. Similarly if the line CF be less than CD, the angle BAF is less than BAD.

Two such lines, therefore, can be drawn; but not more than two.

(3.) The line BE makes a less angle with BA than BF does, when the angle CAE is less than CAF; and the converse.

These conclusions are virtually contained in the preceding argument; and are left for the student to deduce in form.

(4.) Only one straight line HK can be drawn in the plane MN perpendicular to AB.

For that it may be perpendicular to AB, the angles CAH, CAK must be right angles (*Prop.* XIII. *Chap.* II.); and only one line on each side of GC can be drawn perpendicular to AC from the point A. Whence only one line HK, can be drawn in MN perpendicular to AB.

PROPOSITION III.

If a line be oblique to a plane, and about the point in which it meets the plane a circle be described in that plane; and if, moreover, from any fixed point in that line, lines be drawn to the circumference of the circle: then,

(1.) *The least line is that drawn to the intersection of the acute profile trace with the circle; and the greatest, that drawn to the intersection of the obtuse profile trace to the circle;*

(2.) *That which meets the circle at a greater circular distance from the acute trace is greater than that which meets it at the less;*

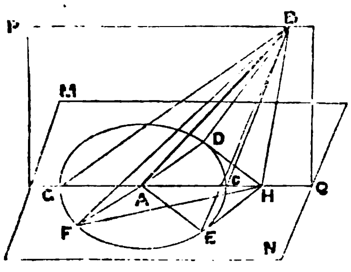
(3.) *Only two such lines can be equal to each other;*

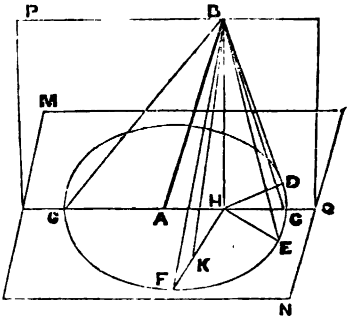
(4.) *The less line makes a greater angle with the plane than the greater line does.*

Let AB be a line inclined to the plane MN, and GDC any circle described in MN about the centre A; and let the trace of the profile plane PQ cut this circle in GC, the acute profile angle being C and the obtuse one G: then

(1.) The least line that can be drawn from B to the circle is BC and the greatest is BG.

For draw the perpendicular BH in the profile plane to the trace GC; and let D be any other point in the circle; and join BD, DH.





Then (*Euc.* iii. 7, 8) HC is less than HD, and HD less than HG; and hence (*Prop.* xi., *Chap.* ii.) BD si greater than BC, and BG than BD. In like manner, BG is greater than BE: that is, BC is the least and BG the greatest line that can be drawn from B to the circle.

(2.) Let F be more remote from C than E is: then BF will be greater than BE.

For join HF.

Then (*Euc.* iii. 7, 8), FH is greater than EH; and hence (*Prop.* xi., *Chap.* ii.) BF is greater than BE.

(3.) Make CD equal to CE; then BD will be equal to BE.

For since HD is equal to HE (*Euc.* iii. 7, 8), BD is equal to BE (*Prop.* xi., *Chap.* ii.).

(4.) The less line BE makes a greater angle with the plane MN than the greater BF does.

For since BF is greater than BE, HF is greater than HE (*Prop.* xi., *Chap.* ii.). Make HK equal to HE and join BK.

Then (*Prop.* xi., *Chap.* ii.) BK is equal to BE, and the angle BKH equal to BEH. But BKH is exterior to BFK, and therefore greater than BFK or BFH. Wherefore, also, BEH is greater than BFH. And these are the profile angles of the lines BE, BF since the planes BFH, BEH are drawn through the common perpendicular BH from the common pointB, and are therefore themselves perpendicular to the plane MN.

Scholium. This proposition might have been enunciated with respect to the edges of a cone whose vertex is B and base the circle GDC.

PROPOSITION IV.

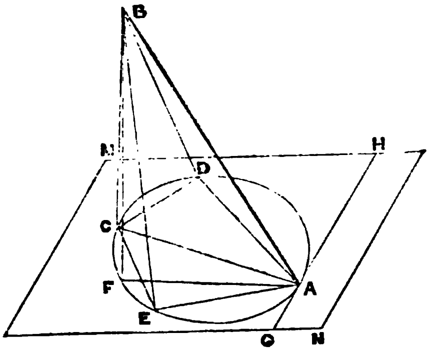
If a line be inclined to a plane, and a plane revolve about the line to meet the plane, and make dihedral angles with it: then,

(1.) *That formed when the trace of the revolving plane is perpendicular to the line itself, is the least;*

(2.) *As the plane revolves from this position towards coincidence (either way) with the profile plane, the dihedral angles increase in magnitude;*

(3.) *Only two positions of the revolving plane, one on each side of the profile plane, can make equal dihedral angles with the given plane.*

Let AB be inclined to MN, and let AC be the profile trace of AB (*Prop.* i., *Schol.* 2); and HG perpendicular to it, that is (*Prop.* xiii., *Chap.* II.) perpendicular to the line AB itself, and let BAE be any other plane through AB: then



(1.) The dihedral angle formed by the plane BAGH with MN is less than any other as BAE.

For, from B draw in the profile plane of AB the line BC perpendicular to AC, and CE perpendicular to AE; and join BE.

Then since CEA is a right angle, CAE is less than a right angle; and hence CA is greater than CE. Wherefore also the angle BEC is greater than BAC (*Prop.* III.): that is, the angle BAC is the least that can be made under the conditions specified.

But since BC is perpendicular to MN and CA, CE perpendicular to lines GH, AE in MN, the lines BA, BE are also perpendicular to these lines (*Prop.* XIII., *Chap.* II). Whence BAC, BEC are the profile angles of the dihedral angles made by the planes BAGH and BAE with the plane MN; and hence the former dihedral angle is less than the latter.

(2.) Let the plane BAF be nearer to the profile plane BAC than the plane BAE is: then the dihedral angle made by BAF with MN will be greater than that made by BAE.

For draw CE, CF perpendicular to AE, AF; and join BE, BF.

Then E, F are in the circumference of a circle whose diameter is AC (*Euc.* III. 31); and by hypothesis AF is nearer to AC than AE is. Whence AF is greater than AE; and consequently CF is less than CE (*Euc.* III. 15). Wherefore, again, the angle BEC is less than BFC. Also, as in the preceding case, these are the profile angles of the dihedral angles made by BAE, BAF with MN. The conclusion therefore follows.

(3.) Let the planes BAD, BAE be such that their traces AD, AE with MN make equal angles with AC: then the planes BAD, BAE make equal dihedral angles with MN.

For draw CD, CE perpendicular to AD, AE; and join BD, BE.

Then since the angles CDA and CEA are right angles, they are in the circle on AC; and since AD, AE make equal angles with the diameter AC (*Constr.*) they are equal. Whence CD is equal to CE; and hence again the angle BDC is equal to BEC.

Also, as before, these are the angles of the dihedral angles enunciated, and the conclusion follows, that two planes can be drawn through AB, which shall make equal dihedral angles with MN.

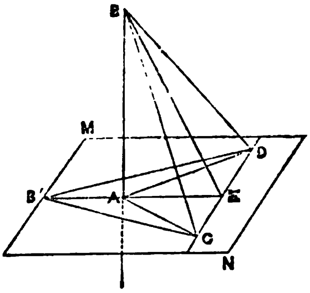
Nor can there be a third plane drawn through AB which shall make a dihedral angle with MN equal to either of these.

For any third plane must be either nearer to the profile plane, and therefore make a greater dihedral angle with MN, or more remote, and therefore make a less, by the preceding case.

PROPOSITION V.

Two points are given in a plane, and likewise a line perpendicular to the plane: assign the point in the perpendicular to which lines drawn from the given points shall contain the greatest angle.

[This is proposed in the form of a problem merely for the sake of brevity in the general enunciation. The conditions will be seen in the several cases of the specific enunciations to be more easily expressible by reference to the figures.]



(1.) Let AB be perpendicular to the plane MN, and C, D two given points in MN; and let the perpendicular AE from A upon the line joining CD meet CD in a point E not beyond the extremities C and D of the line CD: then the angle DAC will be greater than any other angle DBC having its vertex in AB.

For join BE, and then BE is greater than AE, since BAE is a right angle. Make EB′ in EA produced equal to EB, and join B′D, B′C.

Then since B′E, ED are equal to BE, ED each to each, the angle B′ED equal to BED (both being right angles, *Constr.* and *Prop.* XIII., *Chap.* II.); the base B′D is equal to the base BD. And in the same manner it is proved that B′C is equal BC.

Whence since CB′, B′D are equal to CB, BD and the base CD common, the angle CB′D is equal to CBD (*Euc.* I. 8).

But A is within the triangle CB′D; and hence the angle CAD is greater than CB′D; and therefore, also, greater than CBD. That is CAD is the greatest angle that can be formed under the conditions.

In nearly the same way it may be shown that the angles continually diminish as the point B recedes from A in either direction of the line AB from A.

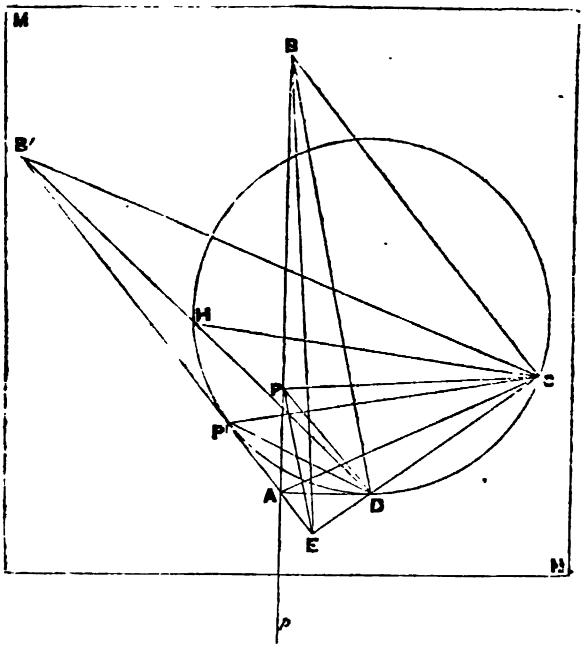
If, also, E had fallen at C or D, the same conclusion is deducible by a slight modification of the reasoning.

(2.) The same general conditions as before, except that E falls in the *prolongation* of CD.

Let AE be the perpendicular on CD produced; produce AE indefinitely, and describe a circle through C, D to touch AE in P′; in AB take a point P, so that EP is equal to EP′: then the angle CPD is greater than any other angle CBD described under the conditions.

For, make EB′ in EA (produced if necessary) equal to EB; and join B′C, B′D.

Then, as before, the angle PED is a right angle (*Prop.* XIII., *Chap.* II.), and therefore equal to P′ED. Whence the two sides and included angle of PED are equal to the two sides and included angle of P′ED; and therefore PD is equal to P′D. In like manner PC is equal to P′C; and therefore (*Euc.* I. 8) the angle CPD is equal to CP′D.



By the same mode of reasoning it is proved that the angle CBD is equal to CB′D.

Now EB′ is a tangent to the circle CDP′ and hence every other point B′ than P′ lies without the circle: whence B′D cuts the circle in some point H. Join HC.

Then the angles CP′D, CHD are equal, being in the same segment; and CHD is greater than CB′D, being the exterior angle of the triangle CHB′. Whence CP′D is greater than CB′D: that is, CPD is greater than CBD.

The same reasoning applies if B be taken between A and P. Whence the angle CPD thus formed is the greatest possible under the conditions on the side of the plane towards B.

There is a point *p*, however, on the other side of the plane MN in AB, such that C*p*D is equal to CPD; viz., at the same distance from A, or such that A*p* is equal to AP.

From this we easily learn that the angles formed at points above P or below *p* continually diminish as their vertices recede from those points respectively upwards and downwards.

Also, that as the vertices approach to A from P and *p*, they also continually diminish to the limit CAD, which is the least angle that can be formed between P and *p*.

And, lastly, if the point of contact P should fall at A, then CAD, instead of being a minimum limit, will become the greatest angle.

Scholium. This proposition is of great importance in orthographic projection, which will be seen under that head. The first part also is the foundation of some of the propositions immediately following this. The proposition itself was first distinctly discussed in the "Cambridge Mathematical Journal," November, 1847.

CHAPTER IV.

TRIHEDRAL AND POLYHEDRAL ANGLES.

PROPOSITION I.

Every two plane angles of a trihedral angle are together greater than the third, and their difference is less than the third angle.

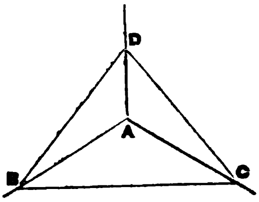
The conditions that may subsist amongst the three angles are the following four:—

(α.) All three angles equal;
(β.) Two equal, and each greater than the third;
(γ.) Two equal, and each less than the third; and
(δ.) All unequal.

Case (α.) Let the three plane angles BAC, CAD, DAB of the trihedral angle at A be all equal: then

(1.) Any two of them BAD, DAC together are double of any one of them BAC, and therefore greater than BAC.

(2.) The difference between any two, as BAD, DAC is nothing; and therefore less than the third angle BAC.



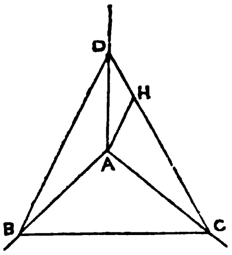
Case (β.) Let DAB, DAC be equal to one another, and each greater than the third angle BAC: then

(1.) The sum of the two BAD, DAC is double of one of them; and each of these is greater than BAC: hence they are both together greater than BAC.

Again, the angles DAC and CAB together are greater than DAC, and hence greater than BAD.

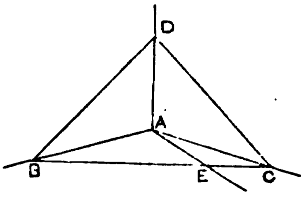
(2.) The angles DAB, DAC being equal, their difference is nothing; and hence less than the third angle BAC.

Again, since DAC is greater than CAB, make CAH in the plane CAD equal to CAB: then the angle DAH is the difference between DAC and CAB. This is less than DAC; and therefore less than the third angle BAD.



Case (γ.) Let two angles BAD, DAC be equal, and each less than the third BAC: then

(1.) The sum of BAD and BAC is greater than BAD; and therefore greater also than the third angle DAC.



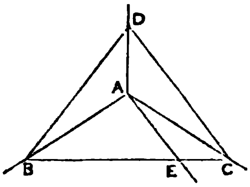
Again, to prove that BAD, DAC are together greater than BAC, make BAE in the plane BAC equal to the angle BAD, and the line AE to AD; draw any line BC through E in the plane ABC, to meet AB, AC in B, C; and join BD, CD.

Then BA, AD are equal to BA, AE and the angle BAD to the angle BAE; hence BD is equal to BE.

But BD and DC are together greater than BC; and hence CD is greater than CE. Whence DA, AC are equal to EA, AC, but the base DC greater than the base EC; and hence the angle DAC is greater than EAC; and consequently the two BAD and DAC together greater than BAE and EAC together, that is than BAC.

(2.) The difference of BAD, DAC is nothing, they being equal angles; and hence less than the third angle BAC.

Again, it has been shown that EAC, the difference between BAC, and one of the other equal angles BAD, is less than DAC the third angle.



Case (δ.) Let all three angles BAD, DAC, CAB be unequal, and let BAC be the greatest of them.

(1.) Take the most unfavourable case to the truth of the proposition, viz., BAD and DAC compared with BAC. Make the angle BAE in the plane BAC equal to BAD, and AE equal to AD; and complete the construction as in the preceding case.

Then by reasoning exactly similar to that in the preceding case, it is shown that BAD and DAC together are greater than BAE and EAC together; that is than BAC.

If either of the angles BAD, DAC be interchanged with BAC, the sum will be greater than before, and the third angle less; and hence, the sum of those two will be greater than the third angle.

(2.) By similar construction to the preceding, and corresponding reasoning, the difference EAC of the angles BAC and BAD is less than the third angle DAC. And this is equally the case whether BAD be the greater or the less of the two angles BAD, DAC.

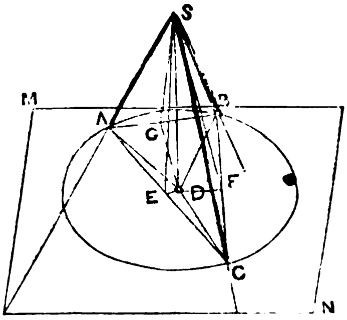
PROPOSITION II.

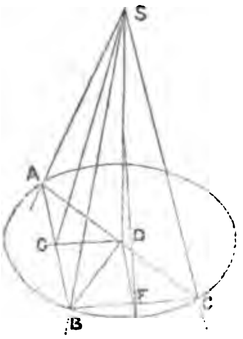
If three points be taken in the edges of a trihedral angle equidistant from the vertex, and a plane be described through these points: and if also a perpendicular to this plane be drawn from the summit, it will be the centre of the circle which passes through the three points in the edges.

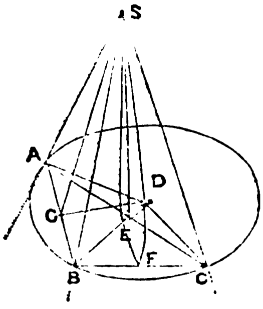
Let SA, SB, SC be the three edges of a trihedral angle, and these distances be all equal; let a plane MN be drawn through ABC, and a perpendicular SD from S to MN: then D is the centre of the circle through A, B, C, in the plane MN.

For join AD, BD, CD.

Then in the triangles SDA, SDB the two sides SD, SA are equal to the two SD, SB; and the opposite angles to SA, SB right angles; therefore the remaining sides and angles are equal, each to each. That is, DB is equal to DA. In like manner DC is equal to DA; and hence A, B, C are equidistant from D, or D is the centre of the circle ABC.







Cor. 1. Each of the edges makes equal angles with the plane ABC: for these angles SAD, SBD, SCD are the corresponding angles of equal triangles.

Cor. 2. Each of the edges makes equal angles with the perpendicular SD.

Cor. 3. If AB, BC, CA be joined, and perpendiculars DG, DF, DE be drawn from D to each of them, these perpendiculars bisect those sides.

Cor. 4. If SE, SF, SG be drawn, the angles SED, SFD, SGD will be the profile angles of the dihedral angles formed by the faces of the trihedral angle with the plane MN.

PROPOSITION III.

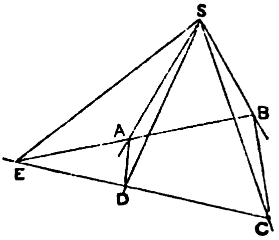
The plane angles of every salient polyhedral angle are together less than four right angles.

(1.) Let the angle be trihedral, and represented by the figures of the preceding proposition. Make the same construction with respect to A, B, C, the plane MN, perpendicular SD, etc.

Then E, F, G will fall *between* the extremities of the lines AC, CB, BA respectively. Whence (*Prop.* v., *Chap.* III.) ADC is greater than ASC, CDB than CSB, and BDA than BSA. Wherefore the three angles at S are together less than the three angles at D together.

But the three angles at D are together equal to four right angles;* and hence the three angles at S are less than four right angles.

(2.) Let the angle at S be tetrahedral, having ASB, BSC, CSD, DSA for its four faces or plane angles: these will together be less than four right angles.



For, produce two of the alternate faces ASB, CSD to meet in SE.

Then (*Prop.* I.) the two angles ASE, DSE are together greater than the angle ASD of the trihedral angle S. EDA. Wherefore also, the two angles BSE, CSE are together greater than BSA, ASD, DSC; and hence again the three angles of the trihedral angle S. BEC at S are greater than the four at S of the tetrahedral angle S. ABCD.

But the angles of S. BEC are less than four right angles by the preceding case; and hence the four angles at S of the tetrahedral angle S. ABCD are together less than four right angles.

Generally. The pentrahedral angle is reduced in the same manner to the tetrahedral, and its plane angles shown to be less; and hence less than those of the tetrahedral angle, and thence again less than four right angles.

PROPOSITION IV.

If from any point within a trihedral angle, perpendiculars be drawn to the faces, these perpendiculars will be the edges of a new trihedral angle, which has the following relations to the original one:—

(1.) *Its edges will be perpendicular to the faces of the original;*

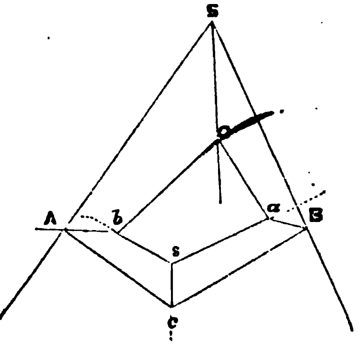
(2.) *Its faces will be perpendicular to the edges of the original;*

(3.) *Its plane angles will be the supplements of the opposite profile angles of the original; and*

(4.) *Its profile angles will be the supplements of the plane angles of the original.*

(1.) This is but a repetition of the hypothesis, for the purpose of enumerating the connected properties *seriatim*.

(2.) Let S. ABC be the original trihedral angle; and from any point *s* within it let perpendiculars *sa*, *sb*, *sc* to the faces BSC, CSA, ASB be drawn: the planes *bsc*, *csa*, *asb* will be perpendicular to the edges SA, SB, SC, respectively.



For let the plane *bsc* cut the planes BSA, ASC in *c* A, *b* A.

Then since *sc* is perpendicular to the plane BSA, the plane *bsc* through it is perpendicular to BSA (*Prop.* XVI., *Chap.* II.). In like manner the plane *bsc* through *bs* is perpendicular to the plane ASC.

Wherefore the plane *bsc*, being

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* In *Fig.* 2, the angle CDA is equal to two right angles; and in *Fig.* 3, the angle CDA is the *reverse* angle.

perpendicular to both the planes BSA, ASC, is perpendicular to SA their common intersection (*Prop.* XVIII., *Chap.* II.).

The two perpendiculars *sb*, *sc* determine, therefore, a plane perpendicular to the edge SA; and in the same manner the other two faces of the trihedral angle *s. abc* are respectively perpendicular to the other two edges of S. ABC; viz., *bsa* to SC and *asc* to SB.

(3.) Since SA is perpendicular to the plane *bsc*, the lines A *b*, A *c* are perpendicular to SA; and hence *b* A *c* is the profile angle of the dihedral angle BSAC. Also (*Prop.* XX., *Chap.* II.), the angles *b*A*c*, *bsc* are supplementary: that is the plane angle *bsc* is the supplement of the profile angle *b* A *c* of BSA, ASC.

In the same manner it may be proved that *csa* is the supplement of the profile angle *c* B *a* of CSB, BSA; and that *asb* is the supplement of the profile angle *a*C*b* of ASC, CSB.

(4.) Since SA, SB, SC drawn from the point S have been proved to be perpendicular to the faces of the trihedral angle *s. abc*, it follows from the preceding case, that the plane angles ASB, BSC, CSA are the supplements of the profile angles *b* C *a*, *a* B *c*, *c* A *b* of the dihedral angles ASC, CSB, of CSB, BSA, and of CSA, ASB; or, merely changing the order, the profile angles of *s. abc* are the supplements of the opposite plane angles of S. BCA.

SCHOLIUM. The trihedral angles thus related are sometimes called *conjugate* trihedral angles, from their interchangeability: sometimes *supplementary* trihedral angles from their properties here proved.

It is easily seen that if *s* be taken at S instead of within the trihedral angle S. ABC, the new trihedral angle would differ from any other formed according to the enunciation of this proposition, it would be in all respects equal to S. *abc*. A direct proof might be made, however, of this fact, from the consideration that the three faces of *any* two trihedral angles thus formed conjugately to S. ABC would be parallel each to each. The details are left for the student to supply.

When S and *s* are coincident, and a sphere is described about S to cut the faces of the two trihedral angles, two spherical triangles are formed, the sides and angles of which have an analogous system of relations. They have also another relation (for which *see* p. 210 of the Spherical Geometry), which causes them to be generally called *polar triangles*: which relation is of great importance in spherical trigonometry.

PROPOSITION V.

The three profile angles of any trihedral angle are together greater than two right angles, but less than six.

(Figure to preceding Proposition.)

Let S. ABC be any trihedral angle, and take a point *s* within it, and construct the conjugate trihedral angle *s. abc* as in the preceding proposition. Then *c* A *b*, *b* C *a*, *a* B *c* are the profile angles of the trihedral angle S. ABC, as already there proved. These three angles are together greater than two right angles.

For the three pairs of angles, *c* A *b* and *csb*, *b* C *a* and *bsa*, and *a* B *c* and *asc*, are together equal to six right angles, each pair being equal to two (*Prop.* XX., *Chap.* II.)

But the three *asb*, *bsc*, *csa* are together (*Prop.* III.) less than four right angles; and hence the other three *a* B *c*, *c* A *b*, *b* C *a* are together greater than two right angles.

Again, the three profile angles are less than six right angles.

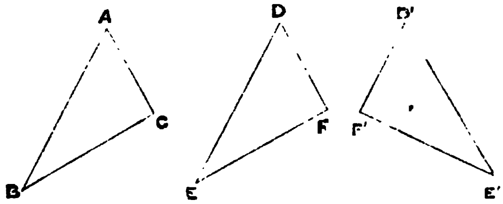
For, since the six angles above enumerated are together equal to six right angles, and that the three *asb*, *bsc*, *csa*, are necessarily of some magnitude, the other three together *a* B *c*, *c* A *b*, *b* C *a*, are deficient from six right angles, by that magnitude whatever it be.

CHAPTER V.

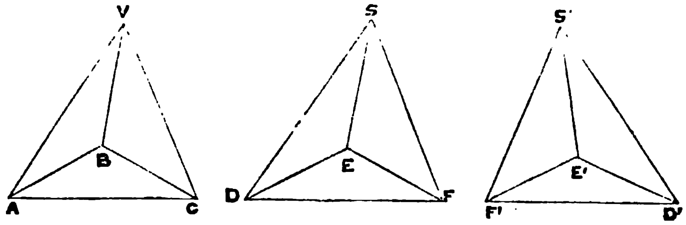
EQUALITY AND SYMMETRY OF TRIHEDRAL ANGLES.

In plane triangles an application of one to the other could generally be made, when the conditions of their structure were three parts in one triangle equal to three corresponding parts in the other. This was rendered possible by the fact of both sides of a plane being alike, so that when supraposition became an effective step, the plane of one triangle might be turned over, and its faces thereby inverted.

In the case of trihedral angles this advantage is lost, and it becomes a necessary condition for supraposition that the three parts of the two figures which are to be made coincident shall follow each other *in the same order*. Thus, in the two triangles ABC, DEF,



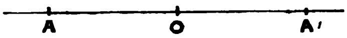
if the sides AB, AC are equal to the sides ED, DF situated in the same order, and the included angles equal, the inference of equality is made by superposition without reversing the plane of the triangle ABC (*Euc.* I. 4.): but if E'D', D'F', were substituted for ED, DF, where the order is inverted, the plane of ABC must be reversed to produce coincidence.



If, now, there be two trihedral angles V. ABC, and S. DEF; and if the two faces, or plane angles AVB, AVC be equal to DSE, DSF and the included dihedral angles equal, they will be capable of coincidence by superposition if the three parts of the one follow each other in *the same order* as in the figure. But if they follow in *a reversed order*, as when S'. D'E'F' is substituted for S. DEF, then equality by coincidence can no longer be inferred; since coincidence itself cannot be made by any change of the faces of any one of the planes concerned.

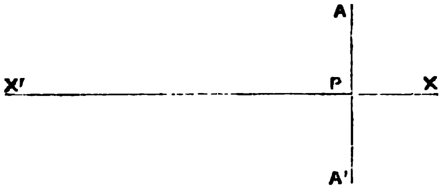
We are then either obliged to have recourse to other methods, or to adopt the principle of *equality by symmetry*.

In general, symmetry may be defined as *similarity of position in reversed directions*, estimated from a point, line, or plane. Thus, if two points A, A', be situated in a straight line, equi-distant from a



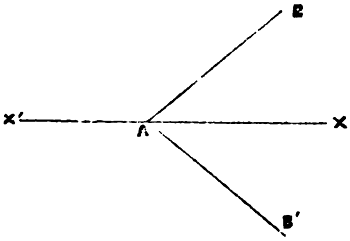
point O in that line, and in *opposite directions*, A and A' are *symmetrical* with respect to the point O.

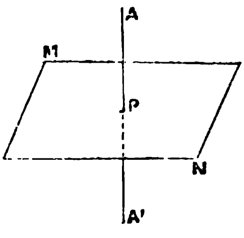
Or, again, if two points A, A' be situated on the opposite sides of a line X'X, and that perpendiculars from them meet X'X in the same



point P, and are likewise equal: then A, A' are *symmetrical* with respect to the line X'X.

Further, if there be two lines AB, AB' drawn from the same point A in the line X'X, making equal angles XAB, XAB', above and below X'X; these lines are symmetrically situated with respect to X'X.



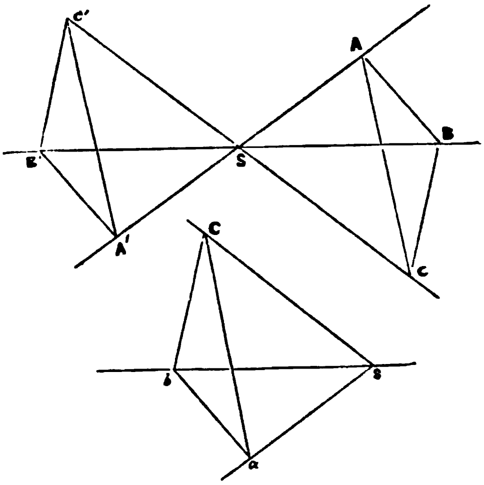


The same holds if for lines we substitute planes: thus, if AA' meet the plane MN at right angles in P, and AP, A'P be equal: then A and A' are symmetrically situated with respect to MN.

Other instances will occur to the student's mind which it will not be necessary to give formally here. It only remains to show the application of the principle to trihedral angles.

(1.) Symmetry with respect to the vertex.

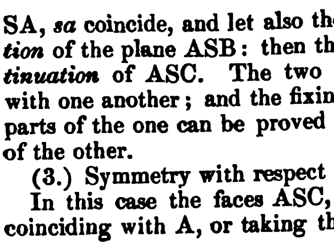
Let S. ABC and *s. abc* be two trihedral angles whose corresponding equal parts lie in contrary directions, or circulate in contrary order. Produce AS, BS, CS, and make SA′, SB′, SC′ equal respectively to *sa*, *sb*, *sc*; and join A′B′, B′C′, C′A′.

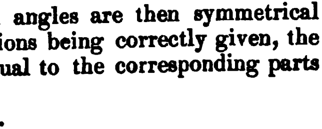


Then the trihedral angle S. A′B′C′ will be equal to *s. abc*, in all respects, if the three fixing conditions of equality be correctly given. It is the business of the investigator to ascertain that this is the case; and to show that the other parts will be equal in magnitude in the two trihedrals whose common vertex is S.

(2.) Symmetry with respect to an edge.

Let S. ABC and *s. abc* be the two trihedral angles which have their pairs of equal magnitudes ranged in reverse order. Make the edges

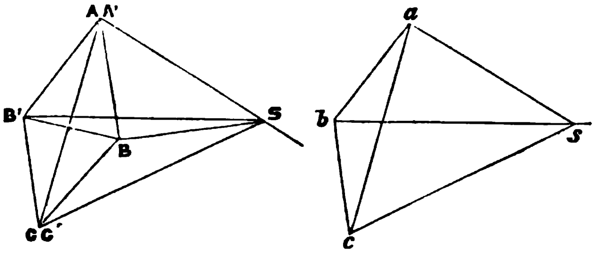




SA, *sa* coincide, and let also the plane *asb* coincide with the *continuation* of the plane ASB: then the plane *asc* will coincide with the *continuation* of ASC. The two trihedral angles are then symmetrical with one another; and the fixing conditions being correctly given, the parts of the one can be proved to be equal to the corresponding parts of the other.

(3.) Symmetry with respect to a face.

In this case the faces ASC, *asc* are placed in contact, the point *a* coinciding with A, or taking the place of A′ in the symmetrical position; and *c* taking the place C or C′. Then B and B′ (that is *b* transposed) is as in the figure; and the fixing conditions of equality being correctly given, it is to be shown that the remaining parts are equal each to each.

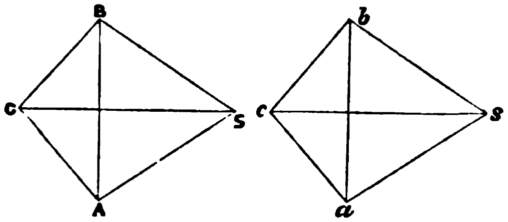


Other forms of symmetry might be given: but these will be sufficient for all the ordinary investigations respecting trihedral angles.

PROPOSITION I.

If two trihedral angles have two faces of the one equal two faces of the other, and their included dihedral angles equal, the remaining face of the one will be equal to the remaining face of the other, and the remaining dihedral angles will be equal, each to each, namely, those to which the equal faces are opposite.

Case 1. Let the two trihedral angles S. ABC, *s. abc* have the two faces BSC, CSA equal to the two *bsc*, *csa*, each to each, and the dihedral angle (CSB, CSA) equal to (*csb*, *csa*); and let the equals succeed each other in the same order; then the faces ASB, *asb*, will be equal; and the dihedral angles (CBS, BSA) and (*cbs*, *bsa*) will be equal, and likewise the angles (CSA, ASB) and (*csa*, *asb*).



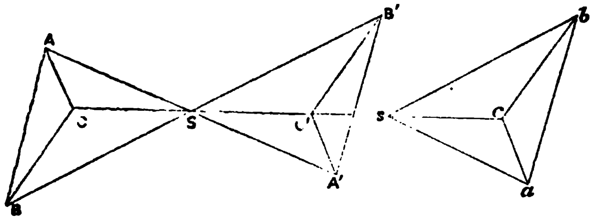
For, let the trihedral angle *s. abc* be applied to S. ABC, so that *s* shall coincide with S, the line *bs* shall coincide with BS, and the plane *csb* with the plane CSB. Then, since the angle *bsc* is equal to BSC, *cs* will coincide with CS.

Again, since the dihedral angle (*csb*, *csa*) is equal to (CSB, CSA) the plane *cas* will coincide with the plane CAS. Also, since the angle *csa* is equal to CSA, the line *as* will coincide with AS, and the plane *asb* with the plane ASB.

Whence, *as*, *sb* coinciding with AS, SB, the angle *asb* is equal to the angle ASB.

Also, since the planes *asb*, *asc* coincide with ASB, ASC, they contain equal dihedral angles; and similarly the dihedral angles (*asb*, *bsc*) and (ASB, BSC) are equal.

Case 2. Let the trihedral angles S. ABC, *s. abc* have the same parts equal as before, but situated in contrary order.



Produce the planes ASC, BSC; their intersection SC' will be in the prolongation of SC. In these planes respectively produce BS and AS, to B' and A', so that SA', SB', SC' may be equal to *sa*, *sb*, *sc*.

Then, since CS and BS are produced, the angle B'SC' is equal to BSC; and this latter by hypothesis, to *bsc*. Whence B'SC' is equal to *bsc*. In the same way A'SC' is equal to *asc*.

Again, the dihedral angle (A'SC', B'SC') is made by the same planes as the angle (ASC, BSC) they being opposite dihedral angles: whence they are equal. Also (ASC, BSC) is equal to (*asc*, *bsc*) by hypothesis; and hence (A'SC', B'SC') is equal to the dihedral angle (*asc*, *bsc*).

Again, since ASB, A'SB' make opposite dihedral angles with the other two faces, those angles are respectively equal each to each; viz., (ASB, CSB) to (A'SB', C'SB') and (ASB, ASC) to (A'SB', A'SC'); and the third face ASB is equal to the third A'SB', since they are opposite plane angles. All the parts of S. A'B'C' are, therefore, equal to the corresponding parts of S. ABC.

But the parts of the trihedral angle S. A'B'C' follow in the same order as those of *s. abc*; and by the first case are respectively equal to them. Wherefore, also, the several remaining parts of S. ABC are equal to the corresponding ones of *s. abc*. That is to say:—

The third face ASB is equal to the third face *asb*; the dihedral angle (ASB, BSC) to (*asb*, *bsc*); and the dihedral angle (BSA, ASC) to (*bsa*, *asc*).

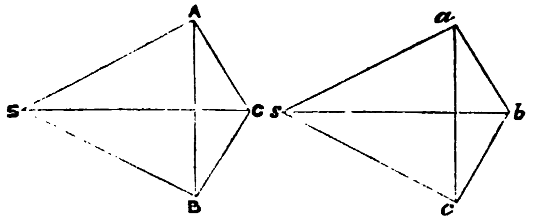
The test of equality laid down by Euclid as the primary one of equality, is capability of *coincidence*. This latter test by *symmetry* is equally valid; and it may be stated generally that, under the circumstances, the two trihedral angles are equal.

The analogous proposition *in plano* is *Euclid* I. 4; and the species of symmetry employed in the proof is the first of those enumerated.

PROPOSITION II.

If two trihedral angles have two dihedral angles of the one equal to two dihedral angles of the other, and likewise their included faces equal: then the remaining dihedral angle of the one will be equal to that of the other; and the remaining faces of the one to those of the other, each to each; viz., those to which the equal dihedral angles are opposite.

Case 1. Let the face ASB be equal to the face *asb*; the dihedral angle (ASB, ASC) to (*asb*, *asc*); and the dihedral angle (BSA, BSC) to (*bsa*, *bsc*) the order of succession being the same in both: then will

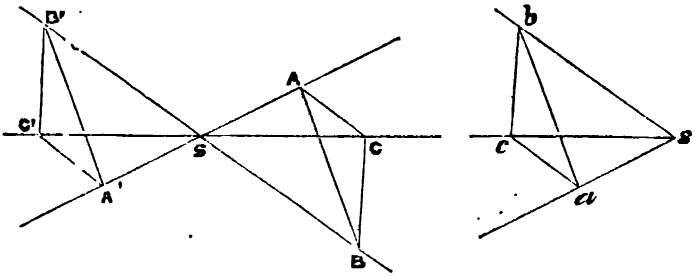


the faces ASC, BSC be equal to the faces *asc*, *bsc*; and the dihedral angle (ASC, BSC) to the dihedral angle (*asc*, *bsc*).

For, place the point S upon *s*, the line SA upon *sa*, and the plane BSA upon *bsa*. Then, since the angles ASB, *asb* are equal, SB will coincide with *sb*.

Again, since the dihedral angle (BSA, ASC) is equal to (*bsa*, *asc*), and the edge SA coincides with the edge *sa*, and the plane BSA with *bsa*, the plane ASC coincides with *asc*; and, similarly, the plane BSC coincides with *bsc*. Wherefore their intersections SC, *sc* coincide; and the trihedral angles wholly coincide. Their corresponding parts, therefore, as detailed above, are equal, each to each.

Case 2. Let the equal parts follow in a contrary order, the lettering of the figures remaining as before.



In the plane ASB produce the lines SA, SB in a symmetrical direction, and likewise produce the planes ASC, BSC; and let SC′ be their intersection.

This will be in the prolongation of SC. Also the angle A′SB′ is equal to its opposite ASB, and therefore to *asb*.

Again the dihedral angle (A′SB′, B′SC′) is opposite to (ASB, BSC), and therefore equal to it; and consequently to (*asb*, *bsc*). Similarly, the dihedral angle (A′SB′, A′SC) is equal to (*asb*, *asc*).

Whence, by the preceding case the trihedral angle *s*. *abc* is equal in all respects to S. A′B′C′.

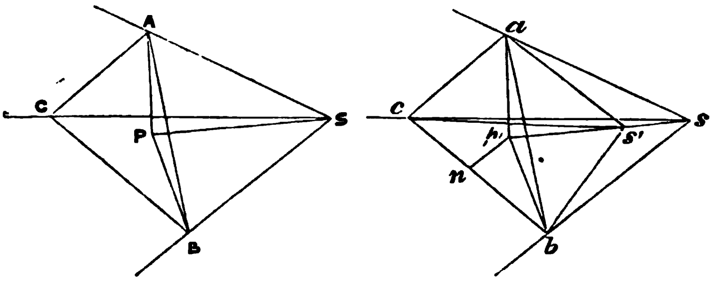
But the dihedral angle (A′SC′, B′SC′) is opposite to (ASC, BSC), and hence equal to it; whence also (ASC, BSC) is equal to (*asc*, *bsc*), that is the remaining dihedral angle of the one to the remaining dihedral angle of the other.

And finally, the faces ASC, BSC are equal to A′SC′, B′SC′; that is, to the faces *asc*, *bsc* respectively.

PROPOSITION III.

If the three faces of one trihedral angle be equal to the three faces of another, each to each: then the dihedral angles of the former will be equal to those of the latter, each to each; viz., those to which the equal faces are opposite.

Case 1. Let the equal parts of the two figures lie in the same order in the two trihedral angles S. ABC, *s. abc*. Make SA, SB, SC, *sa*, *sb*, *sc*, all equal to one another; and draw the planes ABC, *abc*. Through S and *s* draw perpendiculars SP, *sp* to those planes; and join AP, PB, *ap*, *pb*.



Then, since AS, SB, are equal to *as*, *sb*, (*Constr.*) and the angle ASB to *asb* (*Hyp.*), the base AB is equal to *ab*. In a similar manner BC is equal to *bc*, and CA to *ca*. Wherefore the two triangles ABC *abc* are in all respects equal.

Then (*Prop.* II., *Chap.* IV.) P and *p* are the centres of the circles which circumscribe ABC, *abc;* and, therefore, their radii AP, PB, *ap*, *pb* are equal; and the triangle APB is in all respects equal to *apb*.

Let, now, the triangle ABC be applied to *abc*, so that AB, BC coincide with *ab*, *bc*: then the triangle ABC will coincide entirely with *abc*, and the point P with *p* (the bases AB, *ab* coinciding, and the sides being equal each to each). The perpendicular PS, therefore, will coincide with the perpendicular *ps*; and the vertex S will be in the line *ps*.

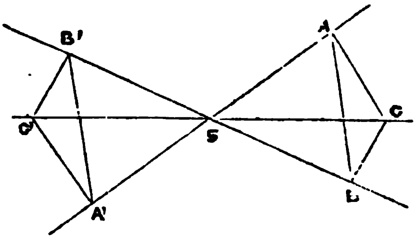
If it be denied that S coincides with *s*, let it be at some other point *s'* in the perpendicular *ps;* and join *as'*, *bs'*, *cs'*.

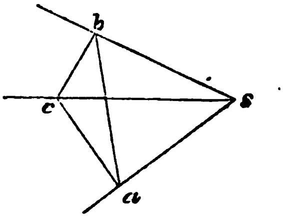
Then since the two sides *cs'*, *bs'*, are equal to CS, BS, and these again to *cs*, *bs*; the two triangles *csb*, *cs'b* have the two sides *cs*, *bs*, equal to the two *cs'*, *bs'*, and the base *bc* common. Whence the angle *cs'b* is equal to the angle *csb*.

But *p* being the centre of the circle about *abc*, the perpendicular from *p* falls *between b* and *c;* and hence (*Prop.* V., *Chap.* III.) the angle *cs'b* is greater or less than *csb*, according as *s'* is in *ps*, or in its prolongation.

These two conclusions are contradictory; and hence the non-coincidence of S and *s* impossible. Whence S coincides with *s*, and the trihedral angles having their parts coincident, those parts are equal each to each; viz., the dihedral angles opposite to the equal faces.

Case 2. Let the equal parts be ranged in reverse order in the two trihedral angles. Produce the three faces of S. ABC: their intersections SA', SB', SC' will be in continuous lines with SA, SB, SC; and hence the angles A'SB', B'SC', C'SA' being opposite to ASB, BSC





CSA, are respectively equal to them. Whence the faces of S. A'B'C' are also respectively equal to those of *s. abc;* and by the former case, the dihedral angles of S. A'B'C' are equal to those of *s. abc*, each to each, respectively.

But the dihedral angle (A'SB', B'SC') is opposite to (ASB, BSC), and therefore equal to it; and hence the dihedral angle (ASB, BSC) is equal to (*asb, bsc*). In the same manner the other dihedral angles are proved to be equal.

CHAPTER VI.

VOLUMES OF SOLIDS.

THIS Chapter is the foundation of the "Mensuration of Solids," and has scarcely any other direct application in mathematics.

The general history of the method is this, as far as regards polyhedrons :—

(1.) To investigate the conditions of equality of two parallelopipeds, or their proportionality to other data.

(2.) To extend the same to triangular prisms—sometimes even to include these amongst the former.

(3.) To divide every polyhedron into parallelopipeds or prisms (most commonly the latter), and then assign a rectangular parallelopiped equal to the sum of them all.

Of the curved solids, notice will be taken hereafter.

It will conduce to convenience of expression to make an unusual definitional distinction here, between a right parallelopiped and a rectangular one.

DEF. 1. A *rectangular parallelopiped* is one which has all its angles right angles and all its faces rectangles.

DEF. 2. A *right parallelopiped* has for one pair of opposite faces any pair of equal parallelograms; but the other four faces rectangles.

This is in reality considering them as *right prisms* upon rectangular bases, and upon parallelogram bases respectively. The view is only changed from its somewhat simplifying the language of the exposition.

There is one important principle to be noticed here, in regard to the equality of solids bounded by plane surfaces. It came before us partially in the last Chapter, but it takes a prominent place in this.

It has been already shown that two trihedral angles may have all the component parts of the one equal to all the corresponding ones of the other, each to each: but that, except these parts succeed each other, in the *same order of rotation*, the two trihedral angles do not admit of supraposition. When they succeed in a reversed order of rotation, they have been called, as already explained, *symmetrical* trihedral angles. We can hardly deny the equality of two things, all the separate elements of which are the same, each in each; merely because their order of succession is from right to left instead of from left to right. It was not, however, necessary to insist upon this in the discussion of *solid angles*, as the terms "equal" and "symmetrical" met the two cases, without any further inference being drawn. Here, however, where the *volume of the space enclosed* is concerned, the case is otherwise. We are compelled *either to prove or assume that the volumes of space comprehended by the boundaries of two symmetrical parallelopipeds or triangular prisms are equal.*

I have never seen a satisfactory proof of this, without the adoption of considerations extraneous to those formally or virtually admitted by Euclid. Neither, though I have long thought about it, have I been more successful than my predecessors.

We have, then, no alternative but to assume as an axiom:—

"If the same triangular prism be equal to one triangular prism and symmetrical to another, these latter prisms contain equal volumes of space."

The case of the parallelopiped can be proved from this, or rather it is a mere corollary. Indeed we are entitled to lay down as a working form of this axiom that:—

Symmetrical triangular prisms and symmetrical parallelopipeds contain equal volumes of space, each to each.

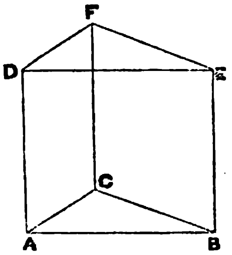
It is deemed proper to draw the student's especial attention to this, rather than by passing it over, to leave him under the belief that his demonstration by supraposition of the equality of symmetrical figures was more rigidly complete than it actually was.

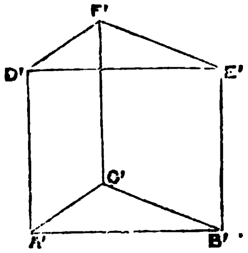
It may be remarked that the *reason* of this difficulty (that supraposition should be universally applicable in the case of plane rectilineal figures, and not in solids bounded by planes) is thus explained. The uniformity of the plane renders a rectilineal figure capable of being *turned over*, so as to enable the figures traced on it to be applied one to the other in the same order though originally given in a reversed one. All the parts of both figures being in the coincident planes, there is no part of the one which may not coincide with a correspondent part of the other. The same, indeed, may be done with any face of the one of two symmetrical prisms and the corresponding face of the other; but this gives the prisms themselves on opposite sides of that common plane, and consequently not in a state of coincidence or supraposition.

Further, if we conceive the figures compared to be successively fitted in a hollow or matrix, it might simplify the conception of this solid supraposition or substitution.

PROPOSITION I.

If two triangular prisms have one solid angle of the one equal or symmetrical to one solid angle of the other, and the three contiguous edges equal, they will have equal volumes.[*]





(1.) Let the triangular prisms ABCDEF and A'B'C'D'E'F' have the three plane angles at A equal to the three plane angles at A', and in the same order; and let the three edges at A be equal to the three edges at A': then these prisms will contain equal volumes.

For if A' be placed on A, A'B' on AB and the plane A'B'C' on ABC, the side A'C' will coincide with AC and the point C' with the point C, since the angle B'A'C' is equal to BAC, and the side A'C' to the side AC.

Also because the trihedral angle at A' is equal to that at A, the line A'D' will coincide with AD; and because A'D' is equal to AD the point D' will coincide with D.

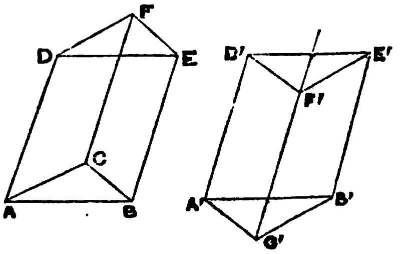
Again, since D'A' and A'C' coincide with DA and AC, and ADEB, A'D'E'B' are equiangular parallelograms, the lines D'E', E'B' will coincide with DE, EB, and the point E' with the point E.

In the same manner the faces F'D'A'C' and FDAC will coincide.

Whence also the face F'D'E' will coincide with FDE, and F'C'B'E' with FCBE.

The boundaries of the two prisms therefore coinciding, they "fill the same space," and are therefore equal.

(2.) Let the two prisms be symmetrical: then by the axiom, they are of equal volume. We give the figure for illustration.



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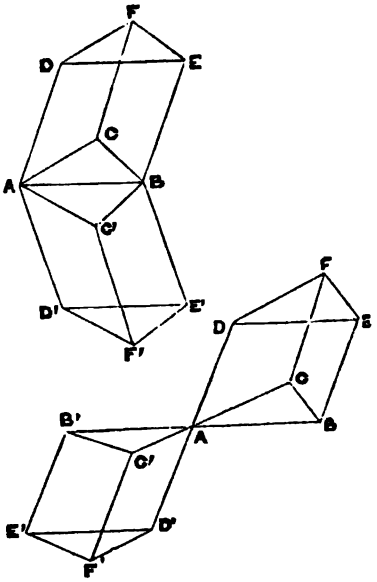
[*] Whatever three conditions are given for defining the equality of the solid angles, the equality of those angles justifies our assumption of any other three that may be more convenient for immediate use. We shall avail ourselves of this fact to avoid circumlocution in the details of our demonstrations.

In these the angle B'A'C' is turned from the side A'B' in a reversed order to that in which BAC is turned from AB. If the face A'D'E'B' were made to coincide with ADEB, the prisms would lie on opposite sides of the plane of the paper (a plane common to both prisms), and coincidence would not take place. The line A'D' would not generally coincide with AD; indeed only when the planes D'A'B', DAB were perpendicular to the planes B'A'C', BAC respectively, or (which implies the same thing) when the angles DAB, DAC, D'A'B', D'A'C' were all equal.



If the base A'B'C' be reversed so as to apply altogether to ABC, the prisms fall on opposite sides of that base; and hence equality by coincidence cannot be inferred. The axiom alone can justify the inference of equality of volumes.

Or again, if the planes of the bases be made continuous, symmetry of the figures results, but no coincidence.

The same conclusions follow in whatever way we dispose of the figures for the purpose of comparison.

PROPOSITION II.

Two parallelopipeds which have one trihedral angle of the one equal or symmetrical to that of another angle in the other; and the edges uniting in those angles equal, each to each corresponding one, will be of equal volume.

The proof is so similar to that in the preceding proposition, that the student ought to draw it up for himself.

PROPOSITION III.

A plane drawn through two opposite edges of a parallelopiped divides its volume into two equal parts.

For it is easily to be shown (by the student) that the parallelopiped is thereby divided into two symmetrical triangular pyramids; and hence the inference by means of the axiom.

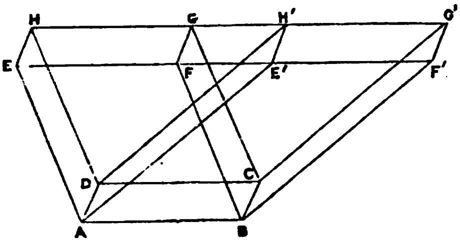
PROPOSITION IV.

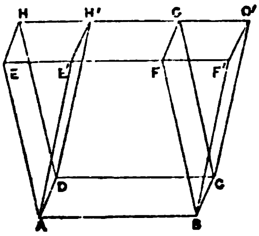
Any plane drawn through the centre of a parallelopiped divides it into two parts which are of equal volume.

This is a slight extension of the preceding one, and it is left for the student to prove that the parts are symmetrical, and hence of equal volume.

PROPOSITION V.

Parallelopipeds on the same base and between the same parallel planes, have equal volumes.

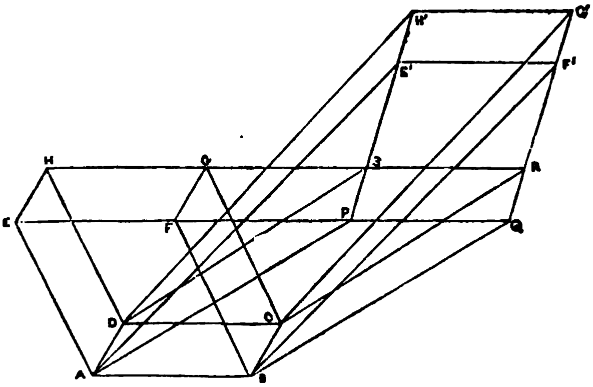




(1.) Let the parallelopipeds AG, AG′ be upon the same base AC, and between the same parallel planes; and first let EF and HG be in the same straight lines with E′F′ and H′G′.

The student must now prove that the triangular pyramid having HDH′ and E′AE for its opposite triangular faces is equal to the triangular prism having G′GC and F′FB for its opposite faces.

The remaining step is analogous to *Euc.* I. 35, and the conclusion is attained.



(2.) Let the parallelopipeds be denoted, in the second place, by the same letters as before, but not have EF and GH coincident with E′F′ and G′H′, nor yet the other pairs of sides coincident.

Produce the sides H′E′, F′G′, HG, FE to meet. They will form a parallelogram PQRS, having its sides parallel and equal to those of the base ABCD; as the student must prove. Whence drawing AP, BQ, CR, and DS, these will be the edges of another parallelopiped on the base ABCD, and between the same parallel planes as the other two.

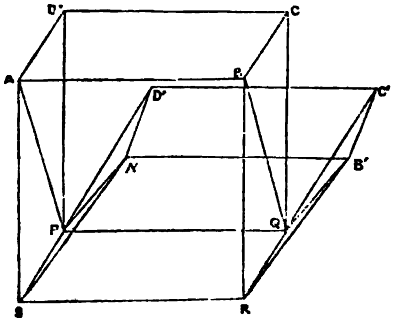
Now by the former case, the parallelopiped AG is equal to AR, by reason of PQ, RS being in the same lines with EF, GH.

And, similarly, the parallelopiped AG′ is equal to AR by reason of PS, QR being in the same lines with H′E′, F′G′.

The parallelopipeds AG, AG′ therefore being each equal to AR, they are equal to one another.

PROPOSITION VI.

Triangular prisms which have one face common, and the edges opposite situate in a plane parallel to that face, have equal volumes.



Let A'PSRB'Q and APSRBQ be two prisms which have the face PQRS common, and the edges AB, A'B' opposite to PR in a plane parallel to PR.

Complete the parallelopipeds SC, SC' on the face PR.

Then it is to be shown that the faces AC, A'C' are in that plane, and hence that the parallelopipeds are equal.

The application of *Prop.* III. will then establish the equality of the prisms.

PROPOSITION VII.

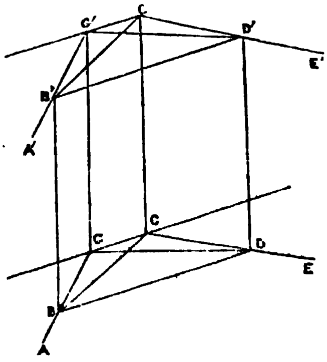
Prisms upon equal polygonal bases and between the same parallel planes are equal to one another in volume.

(1.) Every prism may be transformed into a parallelopiped.

For, let ABCDE, A'B'C'D'E' be corresponding portions of the opposite ends of a prism on the polygonal base. Join BD, B'D', and through C, C' draw CG, C'G' parallel to them; produce AB, A'B' to meet CG, C'G' in G and G'; and join GG'.

Then show that B'G' is equal to BG, and thence infer that GG' is parallel to BB', and therefore also to CC'.

The next step is to show that the plane GCC'G' is parallel to the common face BDD'B'; and hence (*Prop.* VI.) that the prism on the triangular base BGD is equal to that on BCD.

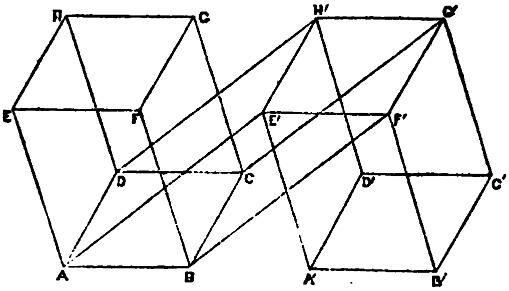


Wherefore, exchanging the former for the latter, the original prism on the polygonal base ABCDE is transformed into one on the base AGDE, having one side less than the original base; and consequently the new prism whilst it is equal in volume to the original one, has one face less.

We may proceed thus till the base is reduced to a quadrilateral, or to a triangle; and whichever we choose we can make a parallelogram equal to it; and consequently the process enables us to form a parallelopiped equal in volume to a given polygonal prism. We may, moreover, make the base of this parallelopiped similar to any specified parallelogram.

(2.) If a second polygonal prism be given, so that their bases have equal areas and the prisms are between the same parallel planes, then the same process may be pursued with the second one, and the bases of both are ultimately reduced to parallelograms equal in all respects.

(3.) We have hitherto supposed the prisms to retain the original parallelism of their edges. We might now suppose one base placed upon the other, and show that the opposite faces are in one plane; and thence infer this equality, and consequently the equality of the original prisms, by means of *Prop.* v. The following, however, is equally simple and more usual, being Euclid's own way, and analogous to his treatment of I. 36.



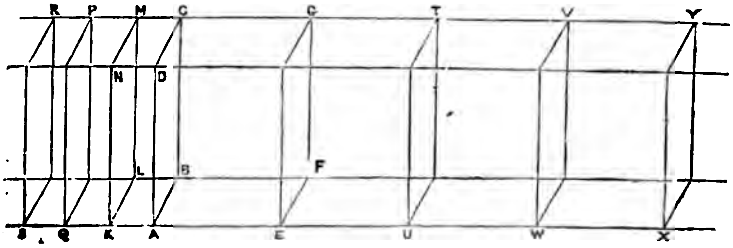
Let ABCDEFGH and A′B′C′D′E′F′G′H′ be the two transformed parallelopipeds between the same parallel planes; and ABCD, A′B′C′D′ the bases equal in all respects, and having the equal sides parallel each to each. Join AE′, BF′, CG′, DH′: then the figure ABCDE′F′G′H′ is to be shown to be a parallelopiped. And by the application of *Prop.* VI., it is to be proved equal to each of the parallelopipeds AG, A′G′; and thence the inference drawn.

Scholium. As the parallelism of the edges may be under any angle whatever with the base of a prism, the edges are for the *purposes of calculation* generally taken perpendicular to the base. This perpendicular is generally called the *altitude of the prism*. When in a problem the conditions are not so given in a direct form, the preliminary step most frequently is, to reduce it to this state from the given relations; and, moreover, to reduce the base of the prism to an equivalent rectangle: in other words, reducing the prism to its equivalent rectangular parallelopiped.

Nor are we obliged to confine this process to the mere practice of calculation; for from what has already appeared, we are equally entitled to do so in our theoretical investigations. The generality of our conclusions is not affected by this restriction, since the preceding theorem reduces (when it may be deemed advantageous) every prism whatever to a rectangular parallelopiped. It will be occasionally adopted here, not to evade any difficulties in the reasoning, but from its simplifying the verbal enunciations of some of the theorems; and from its being at the same time in strict accordance with the ordinary language and notions of practical mensuration.

PROPOSITION VIII.

Prisms of equal altitude are to one another as their bases; and prisms having equal bases are to one another as their altitudes.



(1.) Let the parallelopipeds KC, AG be taken, so that KB, AF their bases be equiangular, and the side AB of the bases and the face AC be common to both; and let them have the same altitude, namely, AD or BC the distance between the two planes KF, NG. Then will the parallelopiped KC be to AG as the base KB is to the base AF.

Take any number of segments in the line AE produced, viz., EU, UW, WX, etc., each equal to AE; from the points U, W, X, etc., draw parallels to EF, meeting BF (in points not lettered), and complete the parallelopipeds ET, UV, WY, etc.

Proceed similarly by taking KQ, QS, etc., each equal to AK, and completing the parallelopipeds QM, SP, etc.

To complete the demonstration, show:—

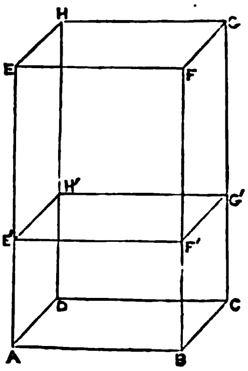
(*a.*) That each of the parallelopipeds ET, UV, WY, etc., is equal to AG; and each of the parallelopipeds QM, SP, etc., is equal to KC.

(*b.*) That whatever multiple AX is of AE, the same multiple is the base BX of the base BE, and the parallelopiped CX of the parallelopiped CE; and similarly for the parallelopipeds AM and AR.

(*c.*) That if AX be greater than AS, the base BX is greater than the base BS, and the parallelopiped AY than AR; but if equal, equal, and if less, less.

(*d.*) Apply *Euc.* v. *Def.* 5, which will establish

par^{d}. AG : par^{d}. AM : : AE : AK
: : base AF : base AL.



(2.) Let the prisms be reduced to rectangular parallelopipeds; as in the figure, where AG and AG′ are the rectangular parallelopipeds equivalent to the two prisms.

In this it is to be shown:—

(*a.*) That the construction is possible.

(*b.*) That the parallelopipeds AG, AG′ being viewed as standing on the bases BG and BG′, they are to one another as BF to BF′.

(*c.*) To modify the expression of this conclusion to agree with the enunciation.

PROPOSITION IX.

All prisms are to one another in a ratio compounded of the ratio of their bases and the ratio of their altitudes.

This is easily deduced from the preceding, and the composition of proportions. It is left for the exercise of the student, as are likewise the two following.

PROPOSITION X.

Parallelopipeds having one trihedral angle of the one either equal or symmetrical to a trihedral angle of the other, are to one another in a ratio compounded of the ratios of the homologous edges terminating in those angles.

PROPOSITION XI.

Similar parallelopipeds are to one another as the cubes of their like linear dimensions;

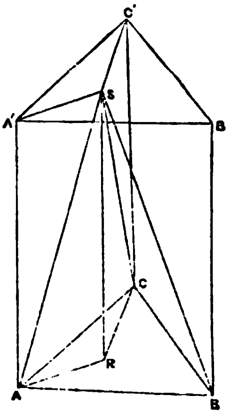
Or again,

As the cubes of any homologous lines of the two figures.

PROPOSITION XII.

Triangular pyramids on equal bases and between the same parallel planes have equal volumes.

The demonstration of this proposition has never been effected in a manner analogous to the preceding ones: namely, by supraposition, the addition and subtraction of figures previously proved to be equal, or the comparison of each with some subsidiary figure. Several forms of proof have been published; but they all involve the same *principle.* We have, therefore, only to select that which appears to be most simple in detail; though even this is sufficiently complicated. We shall break it up as far as possible into distinct steps; dwelling, however, only on those which seem to require more than casual remark to an attentive and intelligent student.



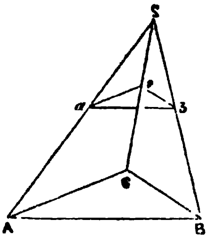
(*a*) *Innumerable prisms may be constituted on the base of a given pyramid, having their opposite ends in a plane through the vertex of the pyramid, and parallel to the base.*

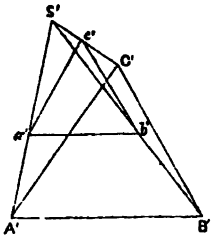
Let S be the vertex, and ABC the base of the pyramid; take any point R within the base ABC, and join RS: then the edges of the prism AA′, BB′, CC′ being drawn parallel to RS, the prism ABC A′B′C′ itself will wholly enclose the pyramid ABCS.*

---

* The same thing will occur if R, instead of being taken *within* the base, be taken *in one of the sides*, or *at an angular point*, the junction of two sides. In complex figures this is sometimes convenient; and in the former case to take the face upon which RS falls for the plane of one face of the prism, and in the other edge with which RS coincides for one of the edges of the prism.

(*b*) *If two pyramids upon equal bases and between the same parallel planes be cut by a third plane parallel to the two former, the sections will be equal to one another.*





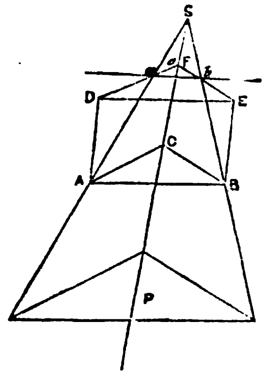
Thus, if ABCS and A'B'C'S' be upon equal bases and between the same parallel planes (that is, having ABC, A'B'C' in one plane, and the vertices S, S' in a plane parallel to it), and *abc*, *a'b'c'* be the sections made in these pyramids by a plane parallel to ABC, A'B'C', they will have equal areas.

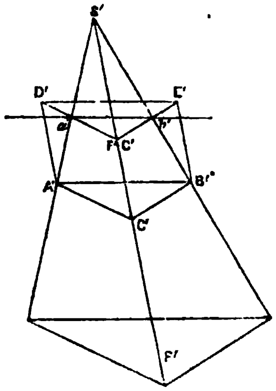
(*c*) *If the parallel sections abc, a'b'c' be made by different planes, their areas will be in the duplicate ratio of their respective distances from the vertices* S *and* S', *reckoned on any convenient parallel lines.*

That is, for example,

area *abc* : area *a'b'c'* :: $Sa^2 : S'a'^2$.

(*d*) *If two pyramids upon equal bases and between the same parallel planes be cut by two other planes also parallel to those planes; and on the sections most remote from the vertices of the prisms as bases prisms be described limited by the other cutting plane; then these prisms will be equal.*





Thus if (P) and (P') be the bases, ABC, A'B'C' and DEF, D'E'F' be the two other planes, the prisms upon the bases ABC, A'B'C' between the same two planes parallel to the bases (P) (P') will be equal.*

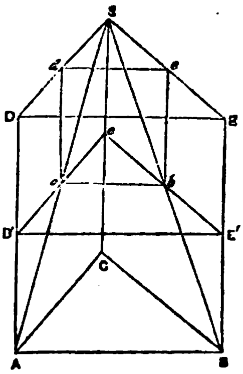
---

* In the figures the edges of the prisms are taken coincident with, and parallel to, the edges SA, S'A' of the pyramids, as described in the note on (*a*) preceding.

(*e*) *If a pyramid between parallel planes be cut by another plane parallel to and equidistant from them, and prisms be erected on the base of the pyramid, and on the section, each limited by the adjacent planes: these two prisms will be less than that erected on the base of the pyramid, and limited by the extreme planes.*

Let SABC be the pyramid, ABC its base, DSE a plane parallel to ABC through the vertex S, and D′E′c a plane equidistant from these two; and let ABCDES be the prism on the base ABC limited by the extreme planes, and ABCD′E′c, $abcde$S the other two, limited each by the plane adjacent to its base: then these two together will be less than the first prism limited by the extreme planes.

Note.—The prisms ABCDES and $abcde$S are similar, and the side of the former is double that of the latter. Their volumes are therefore as $2^3 : 1^3$ or as $8 : 1$; or the smaller one is one-eighth of the larger. Also, the prism ABCD′E′c is one-half of the larger one ABCDES.



Both together, then, are $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ of the larger prism. This ratio is constantly the same from step to step, if we successively bisect as in the next paragraph.

(*f*) *If two pyramids between the same parallel planes, and upon equal bases be cut by planes parallel to these, however numerous and close to each other; and if prisms be constituted on each of these in succession, each prism terminating in the adjacent cutting plane: then the whole of the prisms about one of the pyramids are equal to the whole of those about the other, both separately and collectively.*

This is only a generalized statement of the preceding. It is clear that as we increase the number, and consequently the minuteness of the divisions, these series of prisms will collectively become less and less different in volume from the two pyramids; and it is easy to infer that we may, by the multiplicity of our divisions, render that difference as small as we please. The equality remains the same throughout; and there is no limit that can be set to the closeness of our approximation of the volume in this system of prisms to the volume of the two pyramids.

Here, however, the equality of the pyramids and their systems of dependent prisms is, after all, only approximate and not exact for any *finite* number of deviating planes. We are, therefore, here obliged to introduce a *new principle:*

"Every property which is true *up to* the limit is true *at* the limit."

The proposition is true of the prisms up to their limiting pyramids; and therefore, true of the pyramids, which are the limits of the systems of prisms.

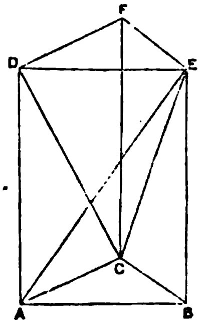
This is precisely the mode in which all such inquiries are conducted, differing only in formal details, in all that relates to figures which do not admit of comparison by superposition, transpositions, or ratios. See several parts of the Mensuration in the first volume of this course, pp. 271, 277.

If the successive divisions be those of continual bisection of the distances of the original planes, it will furnish an interesting exercise of calculation to ascertain the ratio between the prism on the original base and the sums of the successive series of those which result from subdivision. It will be found that they converge towards the ratio of 3 : 1, closer and closer at each successive step; but that they never in any stage actually attain to that ratio exactly. The bisection is only chosen for the ease of calculation; any other mode would have given the same tendent ratio 3 : 1, but the fractions would have been different.

PROPOSITION XIII.

Every pyramid is one-third of the volume of a prism of the same base and altitude.

Let ABCDEF be a prism; draw AE, DC, EC and let planes pass through these lines two and two; thus dividing the prism into three triangular pyramids, whose bases are AEB, AED and DFE having the common vertex C. These are to be shown equal to one another, and hence each of them one-third of the prism itself.



(*a*) Since the face ABED of the prism is a parallelogram, the triangle ABE is equal to the triangle ADE; and hence (*Prop.* XII.) the pyramid ABEC is equal to the pyramid ADEC.

(*b*) Since ABC is equal to DEF, and the planes are parallel, the pyramid ABCE on the base ACB and vertex E, is equal to the pyramid DEFC on the base DEF and vertex C; they being on equal bases and between the same parallel planes.

(*c*) Hence the three pyramids, each of the other two being equal to ABCE, are all equal; and therefore each equal to one-third of the prism of the same base ABC and between the parallels, viz., the altitude or distance of the planes ABC and DEF.

(*d*) What has been proved in reference to the figures in this and the preceding Proposition to figures which are sketched for simplicity as triangular pyramids and prisms, is equally true whatever rectilinear figures constitute their bases; for all prisms can either be transformed to triangular ones as in (*Prop.* VII.) or divided into a series of triangular ones, and reasoned upon collectively.

[In a manner somewhat analogous, but also involving the principle that the circle is the limit of its circumscribing polygons made by successively doubling the number of its sides, the relative volumes of cones, cylinders, and spheres are investigated. Thus, are deduced such theorems as the following:—

(*a*) *Cylinders upon equal bases and between the same (or equidistant pairs of) parallel planes are equal in volume.*

(*b*) *Cones, the same.*

(*c*) *If a cone and cylinder have equal bases and be between the same (or equidistant pairs of) parallel planes, the volume of the cone is one-third of the volume of the cylinder.*

(*d*) *Similar cones, similar cylinders, and spheres have their surfaces as the squares of their homologous lines; and their volumes as the cubes of those lines. In the spheres the diameters are usually taken, but not of necessity, as the homologous lines.*

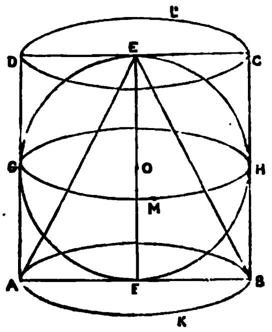
(*e*) *The volumes of two cylinders, or of two cones are in the ratio compounded of the ratio of their bases and the ratio of their altitudes.*

(*f*) *The surface of the sphere is equal to four times the area of its great circle.*

(*g*) *Every sphere is in volume equal to two-thirds of its circumscribing cylinder.*

Hence arises the property of the one, sphere, and cylinder, attributed* to Archimedes, and said to have been modelled on his tomb.

Let AKBCLD be a right cylinder whose height AD is equal to the diameter AB of its base; let also EGFHM be its inscribed sphere, and AKBE a cone on the same base AKB having its vertex in E the centre of the end DLC of the cylinder; then the volume of the cone and sphere together is equal to the volume of the cylinder. For other properties see the Treatise of Archimedes himself in an edition of his works.]



CHAPTER VII.

THE CONE, CYLINDER, AND SPHERE.

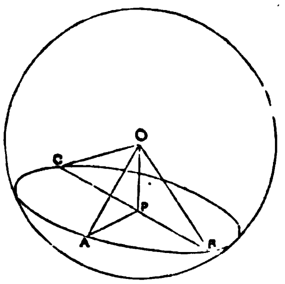
PROPOSITION I.

Every section of a sphere made by a plane is a circle.

1. Let the plane pass through the centre O of the sphere: then all lines drawn from the centre in this plane to the line of intersection being radii of the sphere, they are equal; and hence (*Euc.* I. *Def.* 15) the line of intersection is a circle, of which O is the centre.

2. Let the plane not pass through the centre, but cut the sphere in the figure ABC: then ABC will be a circle.

For from O draw OP perpendicular to the plane ABC, and join PA, PB, PC, etc., and likewise OA, OB, OC, etc.



Then in the right-angled triangles OPA,

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* We say "attributed," because, though Archimedes is the earliest writer in whose works these properties are found, it does not necessarily follow that he was the *discoverer* of every proposition in his Treatise on the Cone and Cylinder, any more than that Euclid was the discoverer of every proposition in his Elements.

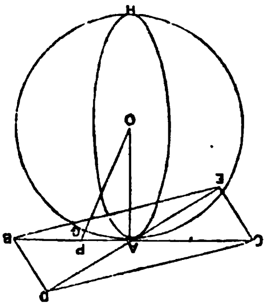
OPB, the two sides OA, OB are equal, and OP common; and hence, also, AP, BP are equal. In the same manner PC, etc., are each equal to AP. Whence ABC is a circle, of which P is the centre (*Euc.* I. *Def.* 15).

COR. 1. If a right cone have its vertex at the centre of a sphere, the sections of the cone and sphere will be circles.

COR. 2. If a cone have its base coincident with a plane section of the sphere, and its vertex at the centre, it will be a right cone.

PROPOSITION II.

If a line or plane be drawn through a point in the sphere, perpendicular to the radius of that point, it will touch the sphere.



1. Let OA be a radius of the sphere, and BC any line perpendicular to OA drawn through A: then AB will touch the sphere.

For let the plane AOBC be drawn, and any other line OP in this plane to meet AB in P, and the circular section made by that plane in G.

Then (as in *Euc.* III. 17.) P is without the circle AGH, and hence AB is a tangent to it.

But AB is in the plane AGH, and hence if it meet the sphere in any other point it must be in this plane; which has been shown to be impossible.

2. Let the plane CEDB be perpendiculor to OA and pass through A: it will touch the sphere.

For since the line OA is perpendicular to the plane CEDB, it is perpendicular to every straight line in it through A; and it has been shown that these straight lines touch the sphere. The plane, therefore, which contains them touches the sphere, since there is no point in that plane through which one of these straight lines does not pass.

PROPOSITION III.

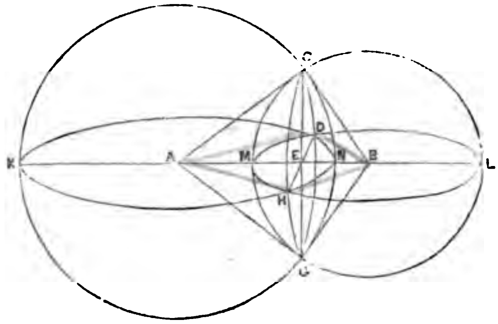
If two spheres intersect each other, their intersection will be a circle the centre of which is in the line joining the centres of the spheres, and whose plane is perpendicular to that line.

Let A, B be the centres of two spheres which cut one another; then the section CDH is a circle. For, draw any two planes through AB to cut the spheres in the circles KCNG, MCLG, and KDNH, MHLD. Join CG meeting AB in E; and join AC, CB, AG, GB, AD, DB, AH, HB, DE, EH.

Then, since CA, CB, are equal to AD, DB, and AB common to the two triangles ACB, ADB, the angles ACB, ADB, are equal; as are likewise the angles CAB, DAB, and the angles CBA, DBA.

Again, since the sides CA, AE are equal to the sides DA, AE, and the included angles equal, the angle DEA is equal to the angle CEA, and the side DE to the side CE.

But the line AB bisects the line CG at right angles in E; and hence the angle DEA is a right angle.

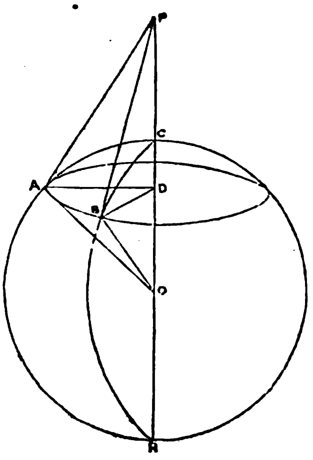


In the same manner, every other line drawn correspondingly to ED is perpendicular to AB. They are all, therefore, in one plane perpendicular to AB.

Also, in the same manner, all these lines are equal to EC; and hence their extremities are in the circumference of a circle, whose centre E is in AB, and whose plane is perpendicular to AB.

PROPOSITION IV.

Through any point without a sphere innumerable straight lines, and likewise innumerable planes, may be drawn to touch the sphere; the linear tangents will be all equal; and the points of contact of the lines and of the planes with the sphere will be a circle.



1. Let O be the centre of the sphere, P a point without it, PA, PB, etc., tangents to the circles CAH, CBH made by planes through PO, and A, B, etc.: then AP, BP, etc., will all be tangents to the sphere.

2. For PA, PB, etc., are perpendicular, to the radii AO, BO, etc.; and hence are tangents to the sphere itself; and as the planes that may be drawn through PO are innumerable, the points A, B, etc., in them are innumerable.

3. Through each of the points A, B, etc., a plane may be drawn perpendicular to AO, BO, etc. These will be tangent planes to the sphere.

Moreover, the lines AP, BP, etc., being perpendicular to AO, BO, etc., they will respectively be contained in the tangent planes at those points. But P is in each of these lines, and hence in each of the planes which contain them: or all the tangent planes pass through P, and are innumerable.

4. Since the right-angled triangles PAO, PBO, etc., have two sides, PO, OA equal to the two PO, OB, the third sides, AP, BP are

equal. Whence all the linear tangents from P to the sphere are equal.

Draw AD perpendicular to PO, and join BD.

Then in the right-angled triangles PAO, PBO, the sides AO, OP are equal to BO, OP, the remaining angles are equal, viz., APO to BPO, and AOP to BOP.

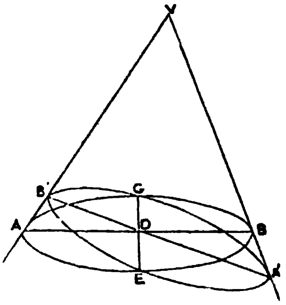
Whence, since AP, PD are equal to BP, PD, and the included angles are equal, the bases AD, BD are equal, and the remaining angles to the remaining are equal. The angle PDB is, therefore, equal to PDA; that is, PDB is a right angle.

Wherefore, AD, BD, etc., are in one plane perpendicular to PD (*Prop.* III. *Chap.* II.), and being all equal, they are in the circumference of a circle whose centre is D in the line PO, and whose plane is perpendicular to PO.

COR. To a given sphere a tangent cone can be drawn having its vertex at any given point without the sphere; and such cone will always be a right cone.

PROPOSITION V.

*If an oblique cone be described on a circular base, then there is another plane parallel to which all the sections are circles.**



Let AGB be the base, and V the vertex of the cone; and let the plane through the axis perpendicular to the base cut the cone in AV, BV, and the base in AB. Draw in the plane AVB a line A'B', making the angle VA'B' equal to VAB, meeting AB in D; and through A'B' draw a plane perpendicular to the plane AVB. The section of the cone by this plane will be a circle.

Now the angles VAB, VA'B' being equal, the points B', A, A', B, are in the same segment of a circle; and hence AD.DB = A'D.DB'.

Again, the planes through AB, A'B' being both perpendicular to the plane AVB, their intersection EG is also perpendicular to that plane, and hence to the lines AB, A'B' themselves.

But by hypothesis and construction, AEB is a semicircle; and hence

$$DE^2 = AD.DB = A'D.DB';$$

whence DE being perpendicular to A'B', and having $DE^2 = A'D.DB'$, the curve A'EB' is also a semicircle.

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* This secondary series of circles are called *sub-contrary* or *antiparallel* to the former: the former name is most frequently used,—the latter the more convenient.

CHAPTER VIII.

GEOMETRY OF THE SPHERICAL SURFACE.

DEFINITIONS.

1. A *sphere* is that figure every point of the surface of which is equally distant from a certain point within it, and that point is the *centre* of the sphere.

2. The distance of any point on the surface from the centre is called a *radius* of the sphere.

3. The *diameter* of the sphere is any line drawn through the centre, and terminated both ways by the surface.

4. A *diametral plane* is any plane drawn through the centre; and its intersection with the surface of the sphere is a *great circle of the sphere.*

5. An *eccentric plane* is any plane which cuts the sphere, but does not pass through its centre; and the intersection of the surface of the sphere by an eccentric plane is called a *less circle of the sphere.*

6. A *tangent plane* to a sphere is a plane which meets the surface of the sphere in one point, and which being produced in all directions, does not cut it.

7. If a perpendicular be drawn through the centre of the sphere to a diametral plane, it intersects the surface in two points, which are called the *poles of the great circle* made by the diametral plane.

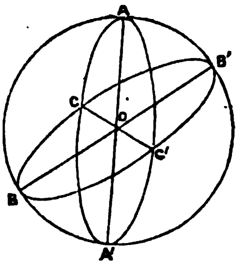
8. If a perpendicular be drawn through the centre to an eccentric plane, it cuts the sphere in two points, which are called the *poles of the less circle.*

9. The poles of a less circle are called the *contiguous* and the *remote poles* respectively, according as the pole is in the less or in the greater segment of the spherical surface made by the plane.

10. When two great circles of the sphere intersect each other, they divide the surface into four portions, called *spherical lunes;* and the enclosed volumes made by the cutting planes and the intercepted surfaces are called *spherical wedges.*

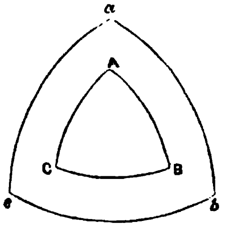
11. When three great circles intersect each other two and two, the portion of the surface enclosed by three contiguous segments of them is called a *spherical triangle*, as ABC in the figure.

12. As the three great circles intersect each other two and two a second time on the opposite sides of the sphere at A′, B′, C′, there will be eight spherical triangles formed by the arcs of the three circles, viz., ABC, A′BC, AB′C, ABC′, and A′B′C′, AB′C′, A′BC′, A′B′C. The first four and the second four sets are called respectively the *primary* and the *symmetrical associated systems* of spherical triangles. The *primary system* is such as to have either AB, BC, or CA in each triangle; and the *symmetrical* as to have either A′B′, B′C′, or C′A′ in each of the triangles which compose the system.



13. Whichever of these triangles be first traced, the entire and completed system will be the same; and hence, any one of them may be taken as the original triangle of the system. It has been usual to consider the central one as the original; but we may readily change the system, as may be easily seen, so that any other of the three shall be the original one. We shall in our investigations, however, consider, except otherwise specially stated, the central one ABC to be the *fundamental triangle of the associated system.* The sides of the other three are either equal to these or supplementary to them: and hence the remaining three triangles will be spoken of as the *supplementary triangles of the associated system.* If we wish to specify which of the supplemental triangles we speak of, it will be conveniently done by mentioning the *side* of the fundamental triangle which is common to the two: thus, BA′C is supplemental with respect to *a*, AB′C with respect to *b*, and BC′A with respect to *c*.

14. If with the angular points A, B, C, as poles, three other great circles be described, their intersections *a*, *b*, *c*, will be the angular points of another spherical triangle *abc*, which is called the *polar triangle* with respect to ABC, the original triangle being called with respect to the polar one, the *primary triangle.*



15. If both the primary and polar system of associated triangles be completed, the systems are called the *primary associated* and the *polar associated* systems.

16. The arcs AB, BC, CA, are called the *sides* of the triangle ABC.

17. By a *spherical angle* is meant the dihedral angle contained by the planes of the circles whose arcs form the boundaries of the lunes on the surface of the sphere.

18. Great circles passing through the poles of a great circle are called *secondaries* to that circle, and the great circle itself is called the *primary* with respect to the secondaries.

19. The *spherical excess* of a triangle is the excess of the three angles of that triangle above two right angles.

20. When two sides or two angles of a spherical triangle are both greater than $\frac{1}{2}\pi$, or both less than $\frac{1}{2}\pi$, they are said to be *of the same affection;* and when one (either side or angle) is greater and the other less than $\frac{1}{2}\pi$, they are said to be *of different affections*, or *of unlike affection.*

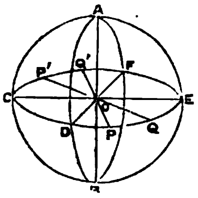
21. A *quadrantal triangle* has one of its sides a quadrant. It is conveniently called a *bi-quadrantal* and a *tri-quadrantal triangle*, when two sides or three sides, respectively, are quadrants.

PROPOSITION I.

The plane of a great circle is cut by the planes of any two of its secondaries in lines which contain an angle equal to the spherical angle formed by those secondaries.

Let CDE be the primary great circle, and ACB, ADB any two of its secondaries (*Def.* 18), cutting the plane CDE in CE, DF: then the angle DOC is equal to the spherical angle DAC.

For, since ACB, ADB are secondaries, their line of section AB is perpendicular to the plane CDE (*Defs.* 7, 18); and hence the planes AOD, AOC cut CDE in lines CO, DO, which are perpendicular to OA (*Pls. and Sols.*, *Chap.* II., *Prop.* XVI.). Wherefore the angle DOC is the measure of the inclination of the planes of the secondaries; and hence again, equal to the spherical angle CAD (*Def.* 17).



Cor. 1. If two great semicircles ACB, ADB, be bisected in C and D, the arc CD of a great circle joining them, is the measure of the spherical angle CAD.

Cor. 2. The pole of a great circle is at the distance of a quadrant from every point of that circle: for since AC, AD, subtend the right angles AOC, AOD, they are quadrants.

Cor. 3. All the secondaries of a great circle are perpendicular to it, for their planes pass through OA, which is perpendicular to its plane; and hence the measure of the angle formed by the circle and its secondary is a quadrant.

Cor. 4. Every great circle perpendicular to another is secondary to it.

Cor. 5. The poles of all the secondaries are situated in the primary.

PROPOSITION II.

The distance of the poles of any two circles is the measure of the spherical angle contained by these.

(See the last Fig.)

Let ACB, ADB be any two circles intersecting in A, B; and let P, Q be their poles respectively: then the arc PQ is the measure of the angle CAD.

For let the circle PQ be drawn to meet the circles ADB, ACB in D and C: then since P is the pole of ACB, PC is at right angles to ACB, and equal to a quadrant (*Prop.* I., *Cor.* 2, 3); and for the same reason QD is at right angles to ADB, and equal to a quadrant. Hence to the secondaries ACB, ADB, CDPQ is the primary; and since CP is equal to DQ, each being a quadrant, we have PQ equal to CD. But since AC, AD are secondaries to CD, the arc CD is the measure of the spherical angle CAD (*Prop.* I., *Cor.* 1); and hence, also, PQ is the measure of that angle.

Scholium. It is essential to remark, that in this demonstration, the poles are taken in continuation of the circle CD in the *same direction*, viz., both in the direction from C towards or beyond D, as P and Q of the figure, or from D towards or beyond C, as Q′ and P′. When the poles are taken *contrarily*, as P, Q′, or P′, Q, their distances measure the *supplementary angles* DAE, and FAC.

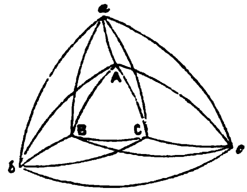
The two cases may be distinguished by the terms *same direction* and *cross direction*.

PROPOSITION III.

If with the angular points of a spherical triangle as poles, three circles be described in cross direction to mutually intersect, a new triangle will be formed such that—

1. *The two triangles shall be mutually polar to each other; and*
2. *The sides of each triangle shall be supplementary to the angles of the other.*

1. With poles A, B, C describe great circles in cross direction with respect to the sides AB, BC, CA of the triangle ABC, mutually intersecting in *a*, *b*, *c*; then ABC, *abc* will be mutually polar to each other.



Since A is the pole of *bc*, the distances A*b*, A*c* are each a quadrant; and similarly B*a*, B*c*, C*a*, C*b*, are also each of them a quadrant. Hence, arranging the order of them differently, *a*B and *a*C are each of them a quadrant; and therefore *a* is the pole of BC. Similarly *b* is the pole of AC, and *c* that of BA.

2. Since B and C are the poles of *ac*, *ab*, taken in *cross direction*, BC is the measure of the supplement of the angle *bac*, formed by *ab*, *ac* (*Prop.* II. *Schol.*), or the side BC is supplementary to the angle *bac*. In the same manner CA and AB are supplementary to the angles *abc* and *acb*.

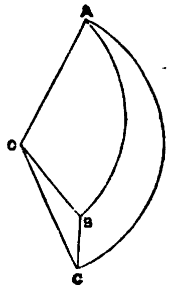
Again, since the triangle ABC is polar to *abc*, the same reasoning will show that *bc* is supplementary to BAC, *ca* to ABC, and *ab* to ACB.

Scholium. This theorem was discovered by Philip Lansberg, and has since his time been extensively employed in trigonometrical research. It is to be recollected, in the enunciation of this theorem, that by a spherical angle is meant its measure, viz., the circular arc intercepted by its sides upon the circle to which they are both secondaries. It must also be carefully remembered that this supplementary property only holds in the two triangles when they are taken in *cross directions*.

PROPOSITION IV.

Any two sides of a spherical triangle are together greater than the third, and the difference of any two is less than the third.

For, let O be the centre of the sphere upon which the spherical triangle ABC is described, and join OA, OB, OC.



Then AOB, BOC, COA, are the plane angles which contain the solid angle at O. Then of these angles the sum of any two is greater than the third, and the difference of any two is less than the third, (*Pls. and Sols.*, *Chap.* IV., *Prop.* I.). Also, these angles are measured by the arcs of the circles which form the spherical triangle ABC; and hence the propocition is true of these arcs, that is, of the sides of the triangle.

PROPOSITION V.

In a spherical triangle, the three sides are together less than an entire circumference, and each side is less than a semicircle.

(See Fig. *Prop.* IV.).

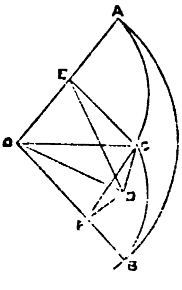
1. For the three angles at O are together less than four right angles (*Pls. and Sols., Chap.* IV., *Prop.* III.), and hence their measures AB, BC, CA are together less than four quadrants: that is, than an entire circumference.

2. Since any two sides AB, BC, are together greater than the third AC, it is obvious that one side. AC, must be less than half the sum of all the three sides AB, BC, CA; that is, by the preceding case, less than a semicircle.

PROPOSITION VI.

In a spherical triangle, if two of the angles be equal, the sides opposite to them are equal, and conversely, if two sides be equal, the angles opposite to them are equal.

1. Let the angles CAB, CBA, be equal; then shall BC be equal to CA.



For, draw CD perpendicular to the plane AOB and in the same plane draw DE, DF, perpendicular to OA, OB, and join EC, FC.

Then (*Pls. and Sols., Chap.* II., *Prop.* XIII.) CE and CF are perpendicular to OA and OB; and hence the angles CED, CFD are equal to the spherical angles at A and B. But (*by hypothesis*) these are equal; and hence in the right-angled plane triangles CDE, CDF we have the angles CED, CFD, equal, and the side CD common; wherefore also EC is equal to CF.

Again, in the right-angled triangles CEO, CFO, we have EC equal to CF and CO common; hence the angles COE. COF, are also equal, and their measures AC, CB, also equal.

2. Let the sides AC, CB be equal, then the angles at A and B will also be equal.

For, make the same construction: then, since the sides AC, CB are equal, the angles AOC, COB are equal; and since OC is common to the two right-angled triangles CEO, CFO, we have EC equal to FC. Wherefore, in the right-angled triangles CED, CFD, we have EC, CD, equal to FC, CD each to each; and therefore the angles CED, CFD also equal: and these are equal to the spherical angles at A and B.

Cor. In every spherical triangle, the greater side and greater angle are opposite.

For, let AC be greater than CB; then the angle COA is greater than COB, and therefore EC greater than CF: wherefore pursuing a train of reasoning analogous to that in the proposition, we have CD common to the triangles CDE, CDF, but EC greater than CF, and hence the angle CFD greater than CED, that is, B is greater than A.

PROPOSITION VII.

Every angle of a spherical triangle is less than two right angles, and all three together less than six right angles.

1. For every dihedral angle is less than two right angles, and hence every spherical angle which has the same measure is less than two right angles.

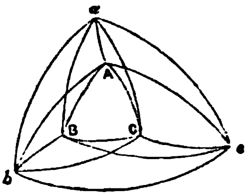
2. Hence, the three together are less than three times two right angles; that is, less than six right angles.

Cor. Hence, contrary to what takes place in plane triangles, there may be two, or even three, of the angles of a spherical triangle either right angles or obtuse angles.

PROPOSITION VIII.

The three angles of any spherical triangle are together greater than two right angles.

For, let ABC be any spherical triangle, and *abc* its cross-polar: and draw the arcs A*b*, A*c*, B*a*, B*c*, C*a*, C*b*.



Then since A is the pole of *bc*, the angle *b*A*c* is measured by *bc*; and since BAC and *bc* are supplementary, the angles BAC and *b*A*c* are together equal to two right angles. In like manner ABC and *a*B*c* are together equal to two right angles, and likewise BCA and *b*C*a* are together equal to two right angles. Hence the six angles ABC, BCA, CAB, *a*B*c*, *b*C*a*, and *c*A*b* are together equal to six right angles.

But the angles *b*A*c*, *c*B*a*, *a*C*b* are measured by the sides of the triangle *abc*, and these are together less than an entire circumference, and the angles themselves, therefore, less than four right angles. Wherefore, the remaining three angles ABC, BCA, and CAB of the spherical triangle ABC are greater than two right angles.

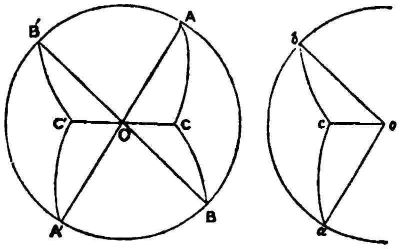
PROPOSITION IX.

If two triangles have three sides of the one equal to three sides of the other, each to each, the angles will be equal which are opposite to the equal sides; and conversely, if the three angles of one be equal to the three angles of the other, the sides opposite to the equal angles will be equal.

1. Suppose the sides to follow in the same order, and let the triangles be *abc*, A′B′C′. Then, as in a plane, the sides *ab*, *bc*, *ac*, will be capable of coincidence respectively with A′B′, B′C′, C′A′, and the conclusion will follow.

2. Suppose them not in the same order, as *abc*, ABC. Then produce the edges OA, OB, OC, to A′, B′, C′, and let the planes of each two cut the spherical surface in B′C′, C′A′, A′B′. Then these are equal to BC, CA, AB, respectively, and hence also to *bc*, *ca*, *ab* respectively.

But the dihedral angles at OA′, OB′, OC′, are equal to those at



OA, OB, OC, since they are vertical to them; and the dihedral angles at *oa*, *ob*, *oc*, are also equal to those at OA′, OB′, OC′, since they are contained by equal plane angles, or those measured by equal arcs. Whence the dihedral angles, that is, the spherical angles of the triangles ABC, *abc* are equal, each to each.

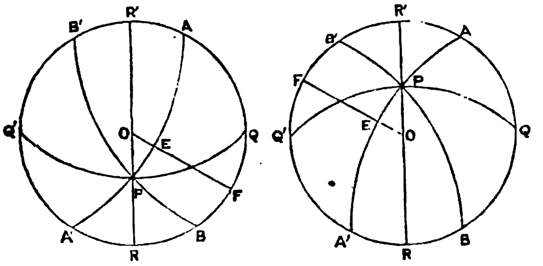
In like manner, by a mere contra-position of the terms of the argument, the converse case may be proved.

SCHOLIUM. By methods correspondent to these, the equalities analogous to those where different combinations of the parts of plane triangles are concerned, may be established.

PROPOSITION X.

Let two great semicircles cut at right angles, then from any point in one of them innumerable pairs of circles may be drawn, so that each pair shall make equal angles with the other semicircle.

Let RPR′, RQR′ be two semicircles intersecting at right angles; then through any point P in the former, innumerable circles may be drawn in pairs, as PA, PB, which shall make, with the latter, angles PAR, PBR equal to one another.



Bisect the semicircle RQR′ in Q; take any two equal arcs AQ, BQ, in RQR′ and draw the circles AP, PB. These will make the angles RBP, RAP equal to one another.

For, produce the circles AP, BP, and draw QP to meet the completion of the circle RQR′, in A′, B′, Q′. Then we have $BR + RA = (RQ - QB) + (RQ + QA) = 2RQ = RQR' = \pi = ARA' = AR + RA'$. Wherefore, $A'R = RB$, and consequently also, $RBP = RA'P = RAP$. Whence, taking AQ, QB, *of any magnitude*, but equal to each other, the angles formed by PA, PB with RQR′ will be equal; which establishes the proposition.

COR. 1. If the point P lie (as in the first figure) between R and the pole O of the circle RQQ′ the angles PAR, PBR will be *acute*, and have as their greatest limit the right angles PRQ′, PR′Q: but if it be beyond O estimated from R (as in the second figure), the angles PAR, PBR will be *obtuse*, and have for their least limit the same right angles PRQ′, PR′Q.

COR. 2. On the hypothesis of the last corollary, PQR will be the least in the former figure, and the greatest in the latter, of all the angles that can be made by circles passing through P and cutting the circle RQR′.

For, draw OEF perpendicular to PA, the one of the circles that passes between PR′ and PQ meeting it in E, and RQR′ in F. Then, since the angle R′PA is acute, it is less than OEP (Fig. 1), and therefore also OE is less than OP; and as QR = OF = $\frac{1}{2}\pi$, we have EF greater than PR. But AA′ is bisected in E, or AE is a quadrant, and at right angles to EF; wherefore A is the pole of EF, and the angle EAF or PAR is measured by EF. Also PQR is measured by PR; wherefore PQR is less than PAR.

In the second figure we may reason similarly; but it will be more simple to deduce the conclusion from the supplements of the angles. Thus, since R′QP (Fig. 2) is less than R′AP, it follows that PQR is greater than PAR.

COR. 3. The circles PA, PB make equal angles with the circle OP or RR′, as is at once obvious.

COR. 4. If from any point P on the surface of a hemisphere bounded by the circle R′QRQ′ circles PR, PB, PQ, PA, PR′, etc., be drawn to meet that circle; then the greatest is that which passes through the pole, and the least is that constituting the remainder of the semicircle; those which pass, one on each side of the greatest, at equal distances from the pole (or which make equal angles on opposite sides with the greatest), are equal, etc., to correspond with *Euc.* III. 7, 8.

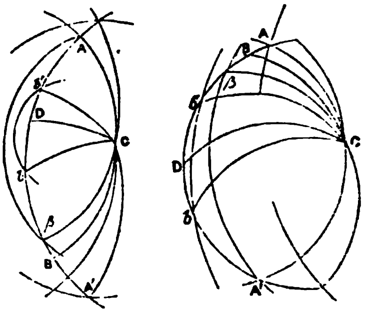
PROPOSITION XI.

Two sides of a triqngle and the angle opposite to one of them being given, it is required to investigate the number of solutions of which a triangle constructed with these data admits.

Let A be the given angle, BC the given side opposite to A, and AC the other given side: it is required to find when the point B can have two positions, and when only one, compatible with the given magnitudes A, a, b.

[In Fig. 1, A is *acute*, and in Fig. 2, A is *obtuse*.]

Produce AB, AC to meet in A′, and from C draw CD perpendicular to ABA′. With pole C, and spherical radius equal to any given value of the side a, describe circles to cut ADA′, which each will do, as for instance, in two points b, b'.



Now, whilst a is less than CD, there will be no intersection, and hence no solution at all; and when a = CD, there will be one solution, the triangle being right angled.

When a is less (in Fig. 1) or greater (in Fig. 2) than either CA or CA′ (but greater than CD), there will be two intersections b and b' (*Prop.* x., *Cor.* 4), and hence two triangles AbC and Ab'C, which have all the data, and therefore two solutions.

When a is equal to, or greater than the less of the two CA, CA′, but less than the other, there will be only *one* solution as AβC, or ABC.

When a is greater than both CA and CA′, the triangle that will be formed will not have the given angle A, but its supplement in the one case; and though formed with A in the other case, one of its sides will be greater than a semicircle, contrary to *Prop.* v.

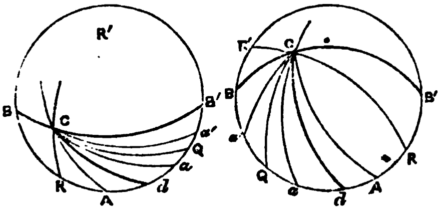
Whilst, therefore, the side a lies between b and $\pi - b$ in value, there will be only one solution: but in all other cases, either two or none.

Cor. The side b and its opposite angle B, in the case of the solution being unique, are of the same affection; as is evident from *Prop.* x., *Cor.* 2.

PROPOSITION XII.

Two angles of a triangle and the side opposite to one of them being given, it is required to investigate the number of solutions of which a triangle constructed with these data admits.

Let B be the given angle to which the given side BC is adjacent, and A the other given angle: it is required to find when two solutions, and when only one, will be compatible with the data.



[In Fig. 1, B is *acute*, and in Fig. 2, B is *obtuse*.]

Draw CR perpendicular to the side AB, and CQ perpendicular to CR: produce BC, BA, to meet again in B′, make Qd = QB′, and draw the great circle Cd.

Then, since CQ, RQ are perpendicular to CR, Q is the pole of CR, and QR is a quadrant; wherefore (*Prop.* x.) CdR = CB′R = CBR. There will hence for any value of the angle A between CdR and CQR be two solutions, as a'BC and aBC; but for any value which shall give to A a position between d and B in Fig. 1, and between d and B′ in Fig. 2, there is only one solution as ABC.

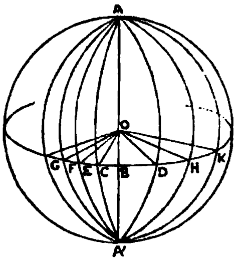
Now the limit to the value of A on the right being CdR, when A = B, and the limit on the left being CBR′ when A = π − B: and the progress from the one to the other being continuous, the condition of the solution being single, is, that A lies between B and π − B. When it is either less in Fig. 1, or greater in Fig. 2, than CQR, there is no solution; and when between the limits B and Q, there will be two solutions. Whilst, therefore, the value of the angle A is between B and $\pi - B$, there will be but one solution: but in all other cases, either two or none.

Cor. The side b and its opposite angle B, in case of one solution, will be of the same affection, as is evident from *Prop.* x., *Cor.* 4.

PROPOSITION XIII.

Spherical lunes and spherical wedges (Def. 10) are to one another as the spherical angles formed at the surface of the sphere by the enclosing arcs of great circles.

Let the planes meeting in the diameter AA′ of the sphere cut the surface in ACA′, ABA′, ADA′. Then the lune contained by ACA′, ABA′, will be to that contained by ABA′, ADA′, as the angle CAB to the angle BAD.



For, let the plane CBD pass through the centre perpendicular to the diameter AA′, cutting the planes which form the lunes in C, B, D. Take in this the arcs CE, EF, FG, etc., each equal to BC, and the arcs DH, HK, etc., each equal to BD; and through the axis AA′, and the several points E, F, G, etc., and H, K, etc., draw planes cutting the surface of the sphere in the great circles AEA′, AFA′, AGA′, etc., AHA′, AKA′, etc.

Then it may be proved by super-position that the lunes BACA′, CAEA′, EAFA′, FAGA′, etc., are all equal; and likewise that BADA′, DAHA′, HAKA′, etc., are all equal. Moreover, that the angles CAB, CAE, EAF, FAG, etc., which are equal to COB, COE, EOF, FOG, etc., and which are subtended by the equal arcs BC, CE, EF, FG, etc., are all equal; as likewise the angles BAD, DAH, HAK, etc., are all equal.

Then reasoning precisely as in *Pls. and Sols., Chap.* II., *Prop.* x., the asserted conclusion follows at once.

In the same manner it may be proved that the *wedges* are proportional to the angles at A.

Cor. 1. The area of any lune is to the surface of the sphere as A to 2π, A being the angle of the lune.

Cor. 2. Also, since the area of the spherical surface is represented by $4\,r^2\varpi$, we have the expression of the area of the lune to A,

$$\text{lune A} = \frac{A}{2\pi} \cdot 4\,r^2\varpi = \frac{A}{\pi} \cdot 2\,r^2\varpi.$$

Of particular cases, notice should be taken of $A = \frac{1}{2}\pi$ and $A = \pi$; in which cases we have, severally,

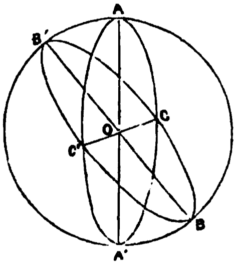
$$\text{lune } 90^\circ = r^2\varpi, \text{ and lune } 180^\circ = 2\,r^2\varpi.$$

It will be advantageous to write the mark) for the word lune; in which case, the expressions above will be briefly expressed as follows:—

$$A^\circ) = \frac{A^\circ}{90^\circ} \cdot r^2\varpi;\ 90^\circ) = r^2\varpi;\ \text{and } 180^\circ) = 2\,r^2\varpi.$$

PROPOSITION XIV.

The area of a spherical triangle is to the surface of the hemisphere as the excess of its three angles above two right angles is to four right angles.



Let ABC be the spherical triangle, and complete the great circles which constitute its containing sides. Then the triangle A'B'C' is equal in all respects to ABC (*Prop.* IX.), and the hemisphere is equal to the four triangles A'BC, AB'C, A'B'C, and ABC or A'B'C'. But we have

$$\triangle\, A'BC = A) - \triangle\, ABC$$
$$\triangle\, AB'C = B) - \triangle\, ABC$$
$$\triangle\, A'B'C = C) - \triangle\, ABC, \text{ or } A'B'C'$$
$$\triangle\, ABC = \triangle\, ABC.$$

Whence by addition we have at once

$$\text{hemisphere} = \pi) = A) + B) + C) - 2\,\triangle\, ABC; \text{ or}$$
$$2\,\triangle\, ABC = A) + B) + C) - \pi.$$

But,

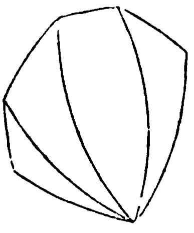
$$A) = \frac{A}{\pi}\,.\,2\,\varpi\, r^2,\ B) = \frac{B}{\pi}\,.\,2\,\varpi\, r^2,\ C) = \frac{C}{\pi}\,.\,2\,\varpi\, r^2,\ \pi) = \frac{\pi}{\pi}\,.\,2\,\varpi\, r^2;$$

and hence, substituting and dividing by 2, we get

$$\triangle\, ABC = \frac{A + B + C - \pi}{2\,\pi}\,.\,2\,\varpi\, r^2;$$

which is identical in signification with the verbal enunciation of the theorem.

Cor. 1. Since the only variable quantity in the expression for the area of the spherical triangle is the *spherical excess*, $E = A + B + C - \pi$, the excess is a correct representation of the area in terms of the unit $\frac{\varpi\, r^2}{\pi}$.



Cor. 2. A polygon of n sides being divisible into $n - 2$ triangles, and the area of each triangle being expressible by means of the spherical excess, we shall have, if Σ signify the sum of all the angles of the polygon,

$$\text{polygon} = \frac{E_1 + E_2 + . E_{n-2}}{2\,\pi}\,.\,2\,\varpi\, r^2$$
$$= \frac{\Sigma - (n-2)\,\pi}{2\,\pi}\,.\,2\,\varpi\, r^2.$$

Cor. 3. If on the surface of the sphere there be described P_p polygons, one of which has n_1 sides, and so on to that which contains n_p sides, (any two or more of these number of sides, however, may be equal); and if Σ_1, Σ_2, ... Σ_p, be the sums of the angles of the several polygons, P_1, P_2, ... P_p:

Then

$$P_1 = \frac{\Sigma_1 - (n_1 - 2)\,\pi}{2\,\pi}\,.\,2\,\varpi\, r^2,\ P_2 = \frac{\Sigma_2 - (n_2 - 2)\,\pi}{2\,\pi}\,.\,2\,\varpi\, r^2, \text{ etc.}$$

And by addition we have $P_1 + P_2 + \dots P_p$

$$= \frac{\Sigma_1 + \Sigma_2 + \dots \Sigma_p - (n_1 + n_2 + \dots n_p - 2p)\pi}{2\pi} . 2\varpi r^2.$$

Cor. 4. If the polygons $P_1, P_2, \dots P_p$, together entirely and exactly cover the sphere, we shall have

$$4\varpi r^2 = \frac{\Sigma_1 + \Sigma_2 + \dots \Sigma_p - (n_1 + n_2 + \dots n_p - 2p)\pi}{2\pi} . 2\varpi r^2.$$

Exercises on the Cone, Sphere, and Cylinder.

1. Equal circular sections are equally distant from the centre of the sphere; and those which are equally distant are equal.

2. The greatest section of the sphere is made by a plane through the centre; and that section which is nearer to the centre is greater than the more remote. And conversely.

3. If two spheres touch each other, the line joining their centres will pass through the point of contact.

4. Two spheres cannot touch one another in more than one point.

5. Through any point on the surface of a cone one tangent plane, and only one, can be drawn: and through any point without the surface, two.

6. If a sphere and right cone touch in two points, they touch in an entire circle.

7. Through any point on the surface of a right cone, innumerable tangent spheres may be described: but only one of them can touch it in more points than one, and this touches in a circle whose centre is in the axis of the cone.

8. If a plane cut a right cone and be not parallel to a tangent plane, then two spheres can be found (one on each side of the plane) which shall touch the plane in points and the cone in circles.

9. If two unequal spheres lie each wholly without the other, two right cones can be described which shall be tangential to the spheres; if the spheres cut each other, only one such cone can be described; and if one sphere lie wholly within the other, no such cone can exist.

10. Planes may be described to touch three spheres, each of which lies wholly without the other two:—how many? Also, investigate the circumstances of any of the spheres cutting or lying within another.

11. Four unequal spheres lie each wholly without the others, and common cones are described to envelope each two: the vertices of these cones are situated in threes upon straight lines, and in sixes in four planes.

CHAPTER IX.

THE CONIC SECTIONS.

THE first volume of this course contains a short discussion on the conic sections by coordinates. So far as the investigation of properties is concerned, the coordinate method is by far the most simple and direct. The elements of the subject are then limited to the equations of the curves, and the equations of their tangents, normals, and diameters. In establishing that method, however, a geometrical property of the conic sections is assumed without proof. It is proposed, therefore, in this chapter, to show how those curves are formed on the surface of a cone, and thence to deduce by geometry some of their most important properties. This course will then contain the elements of the conic sections, both by geometry and analysis.

DEFINITIONS.

1. A *conic section* is the figure formed by the intersection of a plane with the surface of a right cone.* The cone is supposed to admit of indefinite extension, both below the circular base and beyond the vertex, so that the complete conical system is composed of *two opposite cones*, or *two opposite sheets* of the same cone.

2. A plane drawn through the vertex of the cone parallel to the cutting or sectional plane is called the *directing plane*.

3. The section receives different names, according to the position of the directing plane. When this plane touches the cone along one of its edges, the section made by the cutting plane is called the *parabola*. When the directing plane cuts both sheets of the cone, the figure is called the *hyperbola*. In this case there are *two parts* of the curve, which are often called *opposite hyperbolas*. In the investigation of properties, however, those parts are treated only as two branches of the same curve. When the directing plane cuts one sheet only of the complete cone, the figure is called the *ellipse*.†

4. The plane which passes through the axis of the cone, and is perpendicular to the sectional plane, is called the *transverse plane*.

5. The intersection of the transverse and sectional planes is called the *transverse axis* of the conic section. When this line meets the section in two points (as it evidently does in the ellipse and hyperbola), the portion of it intercepted between these points is called the *transverse diameter;* and when the transverse axis meets the section in one point

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* The sections made by a plane with the *oblique* cone, the ellipsoid, and some other surfaces, are also conic sections; but it will be sufficient for the objects of this course to consider only the sections made with the right cone.

† As varieties of these curves, the circle is a case of the ellipse, and is formed when the cutting plane is parallel to the base of the cone (*Pls.* I. 24). The point is also another case when the cutting and directing planes coincide; also when the directing plane of the hyperbolic section coincides with the sectional plane, the section becomes two straight lines.

only (as in the parabola), the portion of it within the curve is called the *principal diameter* or axis of the parabola.

6. The middle of the transverse diameter of the ellipse or hyperbola is called the *centre* of the curve, and a line drawn through the centre at right angles to that diameter is the *conjugate* diameter.

7. *Any* line drawn through the centre of a conic section, and limited by the curve, is called a *diameter* of the conic section. In the case of the parabola, the centre is infinitely distant, and therefore all its diameters are parallel to the principal diameter.

8. The points of section of the diameters with the curve are called the *vertices* of those diameters.

9. A *tangent* to a conic section is a straight line which meets the curve, but being produced, does not cut it.

10. If a tangent and a diameter of a conic section pass through the same point, they are said to be *conjugate to one another;* and any lines drawn parallel to these are said to be conjugate to one another. Of two lines drawn in this way, if both pass through the centre of a conic section, they are said to be *conjugate diameters;* and if one be a diameter, and the other a chord of a conic section, the chord is called an *ordinate* of that diameter. Also the part of the diameter between the vertex and ordinate is called the *abscissa;* and the abscissa and ordinate, when spoken of together, are called *coordinates* of the point or points.

11. The *subtangent* is that portion of the principal diameter of a conic section intercepted between the tangent and the ordinate drawn from the same point in the curve.

12. The *normal* is that portion of the perpendicular drawn to the tangent from the point of contact, which is intercepted between that point and the transverse axis; and the *subnormal* is the portion of the transverse axis intercepted between the normal and the ordinate at the same point.

13. The *parameter,* or (as it is sometimes called) the *latus rectum* of any diameter, is the third proportional to that diameter and its conjugate in the ellipse and hyperbola, and to any abscissa and its ordinate in the parabola.

[This is not a mere arbitrary definition, as might be supposed, but a term suggested by a property of the curve. Its demonstration is given in a subsequent part. By the term parameter in general is meant, some *constant quantity* of the curve. In the case of the circle, for instance, the radius may be considered as the parameter].

14. The *focus* of a conic section is that point in the transverse diameter at which the double ordinate is equal to the parameter of the transverse diameter; and the distance between the centre and focus is called the *eccentricity.*

15. If a third proportional (estimated from the centre in the ellipse and hyperbola upon the transverse axis) be taken to the eccentricity, and the semi-transverse diameter, then the perpendicular to the transverse axis drawn through the extremity of the third proportional is called the *directrix.* In the parabola, the directrix is at the same distance from the vertex that the focus is.

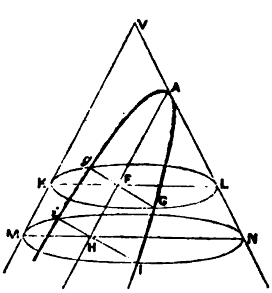
[It will be seen, when we come to the investigation of properties, that the directrix has a remarkable relation to each curve. For instance, the distance of any point in the curve from this line has a *constant ratio* to the distance of the same point from the focus. This property of the conic sections is made the foundation of the coordinate method given in the first volume. The constant ratio has been named the *determining ratio.*]

THE PARABOLA.

PROPOSITION I.

In the parabola, the abscissas are to one another as the squares of their ordinates.

Let MVN be the transverse plane, *i*AI the parabolic sectional plane, AH the transverse axis, KGL, MIN two sections of the cone perpendicular to its axis. Also let KL, MN be the intersections of KGL, MIN with the transverse plane, and GF*g*, IH*i* those with the sectional plane. Then because KGL, MIN are sections perpendicular to the axis of the right cone, they are circles (*Pls.* I. 24); and since the transverse plane passes through the axis of the cone, it passes through the centres of these circles, and therefore the lines KL, MN are diameters of the circles KGL, MIN.



Again, because the plane MVN passes through the axis of the cone, it is perpendicular to the base, and each of its parallel sections KGL, MIN (*Pls.* II. 16). It is also perpendicular to the sectional plane IA*i* (*Def.* 4); hence the two planes KGL, IA*i* being each perpendicular to the plane MVN, their common section G*g* is perpendicular to the same plane (*Pls.* II. 18), and consequently to every line in it, as AH, KL. In a similar way it is shown that I*i* is perpendicular to AH and MN.

Also, since KL is a diameter of the circle KGL, and G*g* is perpendicular to this diameter, G*g* is bisected in F. Similarly I*i* is bisected in H.

Now by the similar triangles FAL, HAN, and the equal lines KF, MH (AH being parallel to VM by *Def.* 3), we have

$$AF : AH :: FL : HN :: KF . FL : HM . HN.$$

But by a property of the circle (*Euc.* III. 35),

$$KF . FL = FG^2, \text{ and } MH . HN = HI^2;$$
$$\text{wherefore } AF : AH :: FG^2 : HI^2.$$

And as AF, AH are obviously the abscissas, and FG, IH the ordinates of the points G and I in the curve, the property is consequently established.

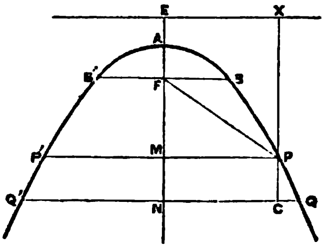
Cor. 1. Let B and P be any two points in the parabola PAP′, and BF, PM their semi-ordinates; then by the proposition,

$$AF : AM :: BF^2 : PM^2,$$
$$\text{or } AF : BF^2 :: AM : PM^2.$$

Hence, if we denote the third proportional to AF and BF, or to

AM and PM, by p, it will be obvious that p will be *constant* for every point in the curve; and therefore we have at any point Q,

$$NQ^2 = p \, . \, AN;$$



that is, if we call p the *parameter* of the principal diameter, *the rectangle under the abscissa of any point in the parabola and the parameter of the axis is equal to the square of the semiordinate of that point.*

This is one of the properties to which reference is made in *Def.* 13.

Cor. 2. Let AF be of such magnitude that $BF = \frac{1}{2}p$; then by *Cor.* 1, $AF = \frac{1}{4}p$. Whence (*Def.* 14), *the abscissa of the ordinate which passes through the focus of a parabola is equal to one-fourth of the parameter of the axis.*

Cor. 3. Let EX be the *directrix* of the parabola PAP′; then (*Def.* 15), $FA = AE =$ (*Cor.* 2) $\frac{1}{4}p$. Wherefore, *the distance between the focus and directrix of a parabola is equal to one-half the parameter of the axis.*

Cor. 4. Let QQ′ be any other ordinate to the axis of the parabola, and PC a line parallel to the axis, meeting the curve in P and the ordinate QQ′ in C. Then by the proposition, $p \, . \, AN - p \, . \, AM = NQ^2 - MP^2 = (NQ + MP)(NQ - MP) = Q'C \, . \, QC$; or $p \, . \, PC = Q'C \, . \, QC$. Hence, *if a double ordinate of the parabola be divided into two segments by a line parallel to the axis, the rectangle under the two segments is equal to the rectangle under the dividing line and the parameter of the axis, the dividing line being limited by the ordinate and curve.*

PROPOSITION II.

In the parabola, the line drawn from the focus to any point in the curve, is equal to the perpendicular drawn from the same point to the directrix.

(See last Figure.)

Let P be any point in the curve, PX a perpendicular on the directrix, and F the focus, of a parabola; then $PF = PX$. For PM being a perpendicular on the axis AM, we have (*Euc.* II. 4),

$$\begin{aligned} PX^2 = EM^2 &= EF^2 + FM^2 + 2EF \, . \, FM \\ &= 2EF \left(\tfrac{1}{2} EF + FM\right) + FM^2. \end{aligned}$$

But by *Def.* 15, and *Prop.* I., *Cor.* 3, $EF = \frac{1}{2}p = 2\,AF$, and also (*Prop.* I.), $p\,AM = MP^2$. Whence $PX^2 = p\,(AF + FM) + FM^2 = p\,AM + FM^2 = PM^2 + FM^2 = PF^2$;

$$\text{or } PX = PF.$$

Cor. *A line drawn from the vertex A perpendicular to the axis, is a tangent at the vertex.*

For any point in this perpendicular is evidently at a greater distance from the focus F than A is, and therefore it is at a greater distance from the focus than from the directrix. Consequently by the proposition, this perpendicular does not meet the curve again; and hence (*Def.* 9) it is a tangent to it.

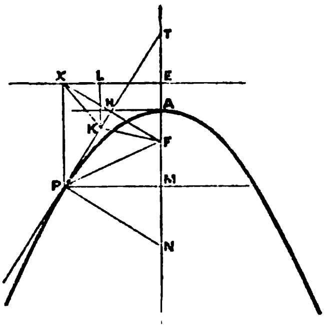
Scholium 1. The property established in this proposition suggests the following simple method of constructing a parabola whose principal parameter is given:—

Make AF = AE = one-fourth of the parameter (*Prop.* I., *Cor.* 2), and in EAF produced, take any point M; with F as centre and radius EM, describe a circle, intersecting a perpendicular PMP′ to EAF produced, in P and P′; then will P and P′ be two points in the curve. Other points are found in a similar way.

Scholium 2. The property itself is the *determining* or *constant ratio*, for the parabola, to which reference is made in *Def.* 15.

PROPOSITION III.

If a line be drawn to bisect the angle made by two lines, one of which is drawn from a point in the parabola to the focus, and the other from the same point perpendicular to the directrix, that line will be a tangent to the parabola at that point.



Let P be a point in the parabola, F the focus, EX the directrix, perpendicular to which PX is drawn; then the line PT drawn to bisect the angle FPX, will be a tangent at P.

For since P is a point in the curve, TP meets the curve; and if it be not a tangent, it will meet the curve again in some other point K. Then, also, K is a point in the parabola. Join KF, and draw KL perpendicular to the directrix, and join XK.

Then because K is a point in the curve, and KL is perpendicular to the directrix, KF = KL (*Prop.* II.); and since by the same *Prop.*, XP = PF, and by hypothesis the angle XPK = FPK, and KP common to the triangles XPK, FPK, the base FK, therefore, is equal to KX. But it has been shown that FK = KL, and therefore KL = KX, the less to the greater, which is impossible. Whence the point K is not in the parabola. And in the same way it may be shown that no other point of PT is in the parabola except P. Hence (*Def.* 9) PT is a tangent to the parabola.

Cor. 1. Since the line XP is parallel to FT, the angle XPT = PTF; but by hypothesis, XPT = FPT. Hence FTP = FPT, and FT = FP; that is, *the point of contact of a tangent to a parabola and the point of intersection of the tangent and axis produced, are equally distant from the focus.*

Cor. 2. Draw the ordinate PM; then AF + AT = FT = FP = PX = EM = EA + AM = AF + AM. Hence AT = AM; that is, *the subtangent of a parabola is bisected in the vertex* (*Def.* 11).

Cor. 3. Draw the normal PN. Then (*Def.* 12), TPN is a right angle, and therefore (*Euc.* VI. 8), TM : MP :: MP : MN, or $TM . MN = MP^2$. But (*Prop.* I., *Cor.* 1), $p . AM = MP^2$; hence $p . AM = TM . MN$. And since by *last Cor.*, TM = 2 AM, consequently MN $= \frac{1}{2} p.$; that is, *in the parabola, the subnormal is equal to half the parameter of the axis.*

COR. 4. From the focus F draw FH perpendicular to the tangent PT, and join A and H. Then because FT = FP (*Cor.* 1), and MA = AT (*Cor.* 2), the triangles PTM and HAT are similar, since the perpendicular FH evidently bisects PT. Wherefore AH is the tangent at the vertex A. Hence, *a perpendicular from the focus of a parabola on any tangent intersects that tangent in the tangent from the vertex.*

COR. 5. Also by *Euc.* VI. 8, FT : FH : : FH : FA; that is, *if a straight line be drawn from the focus of a parabola perpendicular to the tangent at any point, it will be a mean proportional between the distances of the focus from the vertex and the intersection of the tangent and axis.*

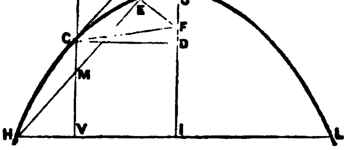
SCHOLIUM 1. From the property of tangents to the parabola established in this proposition, the term *focus* is given to the point F. For by a property of optics, rays of light which proceed parallel to the axis of a parabola and fall on a polished surface whose figure is that produced by the revolution of the parabola about its axis, are reflected to this point.

SCHOLIUM 2. It would be easy to extend this series of properties of the parabola if the limits of this course admitted of such extension. All those properties relative to the principal axis and its coordinates hold for *any diameter* and its coordinates. The following is one of the most important of these.

PROPOSITION IV.

The abscissas of any diameter of a parabola are to one another as the squares of their ordinates.

Let EH be any ordinate to the diameter CV, so that it is parallel to the tangent CT at the point C; CM the corresponding abscissa; AI the axis; and F the focus of a parabola. From E and C draw EG, CD perpendicular to the axis, and from the focus F draw FY perpendicular to the tangent CT; also join FC, and produce EG to meet CV in S.



Then it will be obvious by *Prop.* III., *Cor.* 4, that the triangles FCY, ESM, are equiangular; hence

$$\mathrm{EM}^2 : \mathrm{ES}^2 :: \mathrm{FC}^2 : \mathrm{FY}^2.$$

But (*Prop.* III., *Cor.* 5), since FT = FC,

$$\mathrm{FC} : \mathrm{FY} :: \mathrm{FY} : \mathrm{FA}, \text{ or } \mathrm{FC} : \mathrm{FA} :: \mathrm{FC}^2 : \mathrm{FY}^2.$$

Wherefore, $\mathrm{EM}^2 : \mathrm{ES}^2 :: \mathrm{FC} : \mathrm{FA}$. . . . . . (1).

Again, by the similar triangles CTD, ESM,

$$\mathrm{ES} : \mathrm{SM} :: \mathrm{CD} : \mathrm{DT} :: \mathrm{CD}^2 : \mathrm{CD} \,.\, \mathrm{DT}.$$

And since (*Prop.* I., *Cors.* 1, 2), $\mathrm{CD}^2 = 4\,\mathrm{AF} \,.\, \mathrm{AD}$, and (*Prop.* III., *Cor.* 2), $\mathrm{DT} = 2\,\mathrm{AD}$; we have

$$\mathrm{ES} : \mathrm{SM} :: 4\,\mathrm{AF} \,.\, \mathrm{AD} : 2\,\mathrm{CD} \,.\, \mathrm{AD} :: 4\,\mathrm{AF} : 2\,\mathrm{CD},$$

$$\text{or, } 4\,\mathrm{AF} \,.\, \mathrm{SM} = 2\,\mathrm{CD} \,.\, \mathrm{ES} \quad . \ . \ . \ (2).$$

But (*Prop.* I., *Cor.* 4),

$$4\,AF\,.\,CS = CD^2 - EG^2 = (CD + EG)\,(CD - EG)$$
$$= (CD + EG)\,ES \;.\;.\;.\;.\;.\;.\;.\;.\;(3)$$

Wherefore, taking the difference of (2) and (3), we get

$$(CD - EG)\,ES = 4\,AF\,.\,MC, \text{ or } ES^2 = 4\,AF\,.\,MC.$$

Consequently the proportion (1) becomes

$$EM^2 : 4\,AF\,.\,MC :: FC : FA,$$
$$\text{or, } EM^2 = 4\,FC\,.\,MC.$$

In a similar way it is shown that

$$MH^2 = 4\,FC\,.\,MC.$$

Hence for the same diameter of a parabola, any two abscissas are to one another, as the squares of the corresponding ordinates.

This is the general case of the theorem established in *Prop.* I.

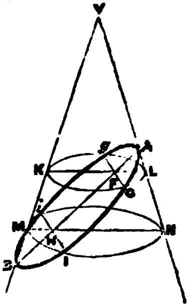
Cor. *Any diameter of a parabola bisects all its own ordinates.*

THE ELLIPSE.

PROPOSITION I.

The squares of the ordinates of the transverse diameter of an ellipse are to one another as the rectangles of their abscissas.

As in the parabola, let AVB be the transverse plane, AGIB the sectional plane; AB the transverse axis, which in this case (*Def.* 5) meets the cone in two points A and B; KGL, MIN, any two sections of the cone perpendicular to its axis. Also let KL, MN, be the intersections of KGL, MIN, with the transverse plane, and GF*g*, IH*i*, those with the sectional plane.



Then as in the case of the parabola (*Prop.* I.), KL, MN are diameters of the circles KGL, MIN, and the transverse diameter AB bisects G*g*, I*i*, perpendicularly in F and H. Hence by the similar triangles AFL, AHN, and KFB, MHB,

$$AF : AH :: FL : HN, \text{ and } FB : HB :: KF : MH.$$

Whence $AF\,.\,FB : AH\,.\,HB :: FL\,.\,KF : HN\,.\,MH.$

But by a property of the circle,

$$FL\,.\,KF = FG^2, \text{ and } HN\,.\,MH = HI^2.$$

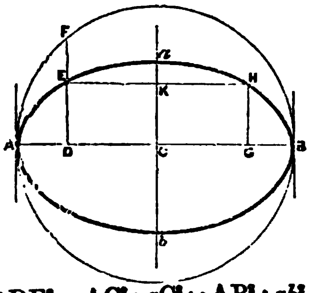
Consequently, $AF\,.\,FB : AH\,.\,HB :: FG^2 : HI^2$;

that is, the squares of the ordinates FG, HI, are to one another as the rectangles of their corresponding abscissas.

It will be obvious that the ellipse is composed of one single and continuous curve, and that the transverse diameter divides it into two equal branches, symmetrically situated on each side of this axis.

PROPOSITION II.

If any point be taken in an ellipse, the rectangle of the abscissas belonging to that point is to the square of the corresponding ordinate as the square of the transverse diameter is to the square of its conjugate.



Let E be any point in the ellipse AaBb, AB the transverse diameter, and ab its conjugate (*Def.* 6); then if ED be drawn perpendicular to AB, the rectangle under AD, DB will be to the square of the ordinate ED as the square of AB is to the square of ab.

For by *Prop.* I.,

$\mathrm{AD} \cdot \mathrm{DB} : \mathrm{AC} \cdot \mathrm{CB} :: \mathrm{DE}^2 : a\mathrm{C}^2$.

But because C is the centre, AC = CB (*Def.* 6); hence $\mathrm{AD} \cdot \mathrm{DB} : \mathrm{AC}^2 :: \mathrm{DE}^2 : a\mathrm{C}^2$, or $\mathrm{AD} \cdot \mathrm{DB} : \mathrm{DE}^2 :: \mathrm{AC}^2 : a\mathrm{C}^2 :: \mathrm{AB}^2 : ab^2$.

Cor. 1. Take p a third proportional to AB and ab; so that $\mathrm{AB} : ab :: ab : p$, then by the property of duplicate ratio, $\mathrm{AB} : p :: \mathrm{AB}^2 : ab^2$. But by the proposition, $\mathrm{AB}^2 : ab^2 :: \mathrm{AD} \cdot \mathrm{DB} : \mathrm{DE}^2$.

Hence $\mathrm{AB} : p :: \mathrm{AD} \cdot \mathrm{DB} : \mathrm{DE}^2$; and similarly for any other point in the curve. This is the property relative to the ellipse, to which reference is made in *Def.* 13. The quantity p is constant when AB and ab are given, and is called the *parameter* of the transverse. Whence this property, *the transverse diameter of an ellipse is to its parameter as the rectangle under the abscissas of any point in the curve is to the square of the ordinate of that point.*

Cor. 2. Produce the ordinate DE to meet the circle described on the axis AB of the ellipse in F. Then by *Euc.* III. 35, $\mathrm{AD} \cdot \mathrm{DB} = \mathrm{DF}^2$. Hence by the proposition, $\mathrm{DF}^2 : \mathrm{DE}^2 :: \mathrm{AC}^2 : a\mathrm{C}^2$, or $\mathrm{DF} : \mathrm{DE} :: \mathrm{AC} : a\mathrm{C}$. A similar property is easily shown to belong to the circle described on ab as a diameter. Hence, *if two circles be described, one on each principal diameter of an ellipse, an ordinate in either circle is to the corresponding ordinate of the ellipse as the diameter to which this ordinate is drawn is to the other diameter.*

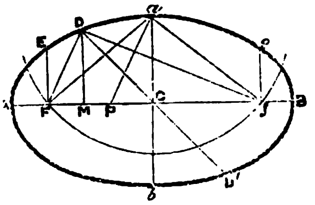
Cor. 3. *Ordinates to either conjugate diameter which intercept equal segments of that diameter from the centre are equal to one another, and conversely, equal ordinates intercept equal segments of the diameter from the centre.*

Scholium 1. By *Def.* 14, the *focus* of the ellipse is that point in the transverse diameter at which the double ordinate is equal to the parameter p (*Cor.* 1) of that diameter; and as *two* such points are evidently on each side of the centre, it will be obvious that in the ellipse there are *two foci* at equal distances from the centre. For a similar reason, there will be *two directrices* (*Def.* 15).

Scholium 2. By means of the property in *Cor.* 2, the curve of an ellipse whose transverse axes are given, may be constructed by points. For if on AB, one of the axes, a circle be described, and in it any ordinate DF be drawn; then since $\mathrm{AC} : a\mathrm{C} :: \mathrm{DF} : \mathrm{DE}$, a point E can be found in the ellipse; and similarly for other points.

PROPOSITION III.

In the ellipse, the square of the distance of the focus from the centre is equal to the difference of the squares of the semi-axes.



Let AB be the transverse diameter of an ellipse, *ab* its conjugate, F and *f* the foci, C the centre, and EF an ordinate at the focus F. Then by *Prop.* II. and *Euc.* II. 5, $CA^2 : Ca^2 :: AF . FB : EF^2 :: CA^2 - CF^2 : EF^2$. But because F is the focus, $EF = \frac{1}{2} p$ (*Def.* 14), and therefore by *Def.* 13 $AC^2 : aC^2 :: aC^2 : \frac{1}{4} p^2$ or EF^2. Hence, $CA^2 - CF^2 = aC^2$, or $CF^2 = CA^2 - aC^2$; that is, the square of CF, the distance between the centre and focus F, is equal to the difference of the squares of the semi-axes AC, *a*C. And similarly for the distance C*f*.

Cor. 1. By the proposition $CF^2 = CA^2 - Ca^2$, or $CF^2 + Ca^2 = CA^2$. But $CF^2 + Ca^2 = Fa^2$; hence $CA^2 = Fa^2$, or $CA = Fa$. Wherefore, *the distance of either extremity of the conjugate diameter from the focus is equal to the semi-transverse diameter.*

Cor. 2. Since $Ca^2 = CA^2 - CF^2 = AF. FB$; hence, *the rectangle under the segments of the transverse diameter made by one of the foci is equal to the square of the semi-conjugate axis.*

PROPOSITION IV.

If in the ellipse a fourth proportional be taken to the semi-transverse axis, the eccentricity, and the distance of an ordinate to any point in the curve (all estimated from the centre along the transverse axis); then the extremity of this fourth proportional will divide the transverse axis into two parts which are equal respectively to the focal distances of that point.

(See the Fig. of last Prop.)

Let D be any point in the ellipse A*a* B*b*, F and *f* the foci, and CP a fourth proportional to CA, CF and CM (DM being the ordinate to the point D); then the segments AP, BP of the axis AB will be respectively equal to the focal distances FD, *f*D, drawn from the foci to the paint D.

For by construction,

$CA^2 : CF^2 :: CM^2 : CP^2$; hence since (*Prop.* III.), $CA^2 - CF^2 = Ca^2$,

$$CA^2 : Ca^2 :: CM^2 : CM^2 - CP^2.$$

But (*Prop.* II.), $CA^2 : Ca^2 :: CA^2 - CM^2 : MD^2$.

Whence, $CM^2 : CA^2 - CM^2 :: CM^2 - CP^2 : MD^2$.

From this we readily get,

$$CA^2 : CM^2 :: CM^2 - CP^2 + MD^2 : CM^2 - CP^2,$$

$$\text{or } CA^2 : CM^2 - CP^2 + MD^2 :: CM^2 : CM^2 - CP^2$$

$$:: CA^2 : Ca^2.$$

Hence $Ca^2 = CM^2 - CP^2 + MD^2 = CD^2 - CP^2$; that is,

$$CD^2 = Ca^2 + CP^2 = Pa^2.$$

Now (*Euc.* II. 12)

$$\mathrm{FD}^2 = \mathrm{FC}^2 + \mathrm{CD}^2 - 2\,\mathrm{FC}.\mathrm{CM};$$

but as has been shown,

$\mathrm{CD}^2 = \mathrm{P}a^2 = \mathrm{PC}^2 + \mathrm{C}a^2$; also (*Constr.*), CA : CF : : CM : CP, or FC . CM = CA . CP, and (*Prop.* III.), $\mathrm{FC}^2 = \mathrm{CA}^2 - \mathrm{C}a^2$.

Wherefore $\mathrm{FD}^2 = (\mathrm{CA} - \mathrm{PC})^2$, or FD = CA − CP = AP.

In a similar way it is shown that BP = *f* D.

Cor. 1. Since FD = AP, and *f* D = BP; hence FD + *f* D = AB; that is, *the sum of the two lines drawn from the foci to any point in the ellipse is equal to the transverse diameter.*

Cor. 2. With F as centre and *f* D radius, describe a circle to intersect the ellipse in D′, *below* AB; then it follows from the proposition that the distance of D′ from *f* is equal to FD. Wherefore if the points F, D′ and *f*, D′ be joined, the figure FD*f*D′ will be a parallelogram; and, therefore, since the diagonals of a parallelogram bisect one another, DD′ passes through C the middle of F*f*; that is, DD′ is a diameter (*Def.* 7), hence *all diameters of the ellipse are bisected in the centre.*

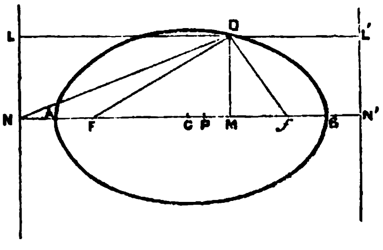
Scholium. From this proposition is deduced a very simple method of tracing an ellipse by points. For in the transverse axis AB, take the foci F, *f*, and *any point* P; with both centres F, *f* and radii AP, BP, describe circles; these will intersect in *four* points, which are evidently four points in the curve. Proceed similarly with another point taken in AB, and get four other points in the curve, and so on to any extent.

PROPOSITION V.

The distance of the directrix from any point in the ellipse has to the distance of the focus from the same point a constant ratio.

Let NL be the directrix with respect to the focus F, and N′L′ that with respect to *f*; then the ratio of FD to DL, or the ratio of *f*D to DL′, is equal to the constant ratio of CF to CA (C being the centre), wherever the point D is taken in the curve.

Take, in the axis produced, CF : CA : : CA : CN, and draw NL perpendicular to CN; then NL is the directrix (*Def.* 15).



Also take, as in *Prop.* IV., CP a fourth proportional to CA, CF, and CM, the distance of the ordinate DM from the centre. Then we have

CF : CA : : CA : CN, and CF : CA : : CP : CM.

Hence, CA : CN : : CP : CM, or CA : CP : : CN : CM.

Wherefore we readily get by composition,

AP : CA : : MN : CN, or since AP = FD, by *Prop.* IV., and MN = DL; FD : DL : : CA : CN : : CF : CA.

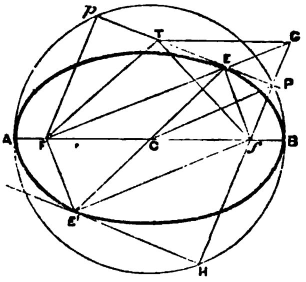
Similarly, *f*D : DL′ : : CF : CA.

Scholium. The *constant ratio* investigated in this proposition is the property referred to in *Def.* 15, with respect to the ellipse.

PROPOSITION VI.

The straight line which bisects the angle adjacent to that which is contained by two straight lines drawn from any point in the ellipse to the foci is a tangent to the curve at that point.

To E any point in the ellipse, let the straight lines FE, fE, be drawn from the foci; then the straight line EP which bisects the angle fEG which is adjacent to FEf, is a tangent to the curve.



Take T any other point in PE produced; and make EG = Ef, and join Tf, TG, and fG, meeting EP in P. Because Ef = EG, and EP is common to the triangles EfP, EGP, and the angles fEP, GEP are equal, these triangles are equal, and fP = PG; and hence fT = TG. Wherefore FT + fT = FT + TG; but FT + TG is greater than FG, that is, greater than FE + Ef, or AB. Consequently FT + fT is greater than AB. But if T be a point in the curve, FT + fT = AB (*Prop.* IV., *Cor.* 1), which is impossible; and therefore T is not in the curve. In the same way it may be shown that no other point in PET but E is in the curve; that is, PET is a tangent at the point E.

Cor. 1. Since the angle fEP = GEP, it must also be equal to FET which is the vertical angle of GEP. Hence *a tangent to the ellipse makes equal angles with the straight lines drawn from the point of contact to the foci.*

Cor. 2. Produce EC to meet the curve again in E', and draw the tangent HE'; then EE' is a diameter (*Def.* 7), and it is bisected (*Prop.* IV., *Cor.* 2), in C. The line Ff is also bisected in C; hence FEfE' is a parallelogram, and the angle FEf = FE'f. Now (*Cor.* 1) since the angle TEF = PEf, each of these angles is half the supplement of FEf; and for the same reason the angle HE'f is half the supplement of FE'f; and since FEf = FE'f, the angle TEF = HE'f. Also, because EF is parallel to E'f, the angle FEC = CE'f; consequently the whole angle TEE' = HE'E, and therefore PET is parallel to HE'; that is, *the tangents at the opposite extremities of any diameter are parallel to one another.* The convex of this is also true, viz., that *parallel tangents are at the extremities of the same diameter.*

Cor. 3. Let fP be a perpendicular from the focus f on the tangent at the point E; and let PG = Pf, as before, so that FEG is a straight line equal to AB; join CP. Then because FC = fC, and fP = PG, the triangles FGf, CPf, are similar. Whence FG : CP : : Ff : Cf : : 2 : 1; hence CP = ½ FG = ½ AB = CB. Wherefore the point P is in a circle described on AB as a diameter. Similarly, the perpendicular Fp on the same tangent meets that tangent in the circle on AB. Consequently, *the perpendiculars drawn from the foci on any tangent to the ellipse meet that tangent in a circle described on the transverse diameter.*

Cor. 4. Since the tangent at E' (*Cor.* 2) is parallel to that at E, the perpendicular from f on the tangent at E' will meet that tangent in its point of intersection H with the circle on AB, so that $f\mathrm{P}$, $f\mathrm{H}$, are in the same straight line. Wherefore (*Eu*. III. 35), $\mathrm{A}f . f\mathrm{B} = \mathrm{P}f . f\mathrm{H}$. But the triangles $\mathrm{F}p\mathrm{E}$, $f\mathrm{HE}'$ are evidently equal in all respects, and therefore $f\mathrm{H} = \mathrm{F}p$; consequently $\mathrm{A}f . f\mathrm{B} = \mathrm{P}f . \mathrm{F}p$. Whence, since the rectangle under $\mathrm{A}f$ and $f\mathrm{B}$ (*Prop.* III., *Cor.* 2) is equal to the square of the semi-conjugate diameter, *the rectangle under the perpendiculars from the foci of an ellipse upon any tangent is equal to the square of the semi-conjugate diameter.*

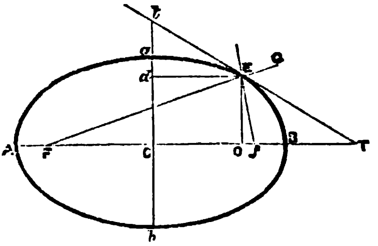
COR. 5. *A perpendicular to the transverse axis at either of its extremities is a tangent to the ellipse; also a perpendicular to the conjugate axis at either of its extremities is a tangent to the curve.*

PROPOSITION VII.

If a tangent and an ordinate be drawn from any point in the ellipse to meet the transverse axis, the semi-transverse axis will be a mean proportional between the distances of the two intersections from the centre.

For let ET be a tangent to the ellipse AEB at the point E, and ED an ordinate at the same point to the transverse diameter AB; then because ET (*Prop.* VI.) bisects the angle $f\mathrm{EG}$,

$$\mathrm{FT} : \mathrm{T}f : : \mathrm{FE} : \mathrm{E}f;$$



hence $\quad \mathrm{FT} + \mathrm{T}f : \mathrm{FT} - \mathrm{T}f : : \mathrm{FE} + \mathrm{E}f : \mathrm{FE} - \mathrm{E}f,$

or, $\quad 2\,\mathrm{CT} : \mathrm{F}f : : 2\,\mathrm{CB} : \mathrm{FE} - \mathrm{E}f \quad . \quad . \quad . \quad . \quad . \quad (1).$

But $\mathrm{FE}^2 - \mathrm{E}f^2 = \mathrm{FD}^2 - f\mathrm{D}^2$, or $(\mathrm{FE} + \mathrm{E}f)(\mathrm{FE} - \mathrm{E}f) = (\mathrm{FD} + f\mathrm{D})(\mathrm{FD} - f\mathrm{D})$; that is, since $\mathrm{FE} + \mathrm{E}f = 2\,\mathrm{BC}$, $\mathrm{FD} + f\mathrm{D} = \mathrm{F}f$, and $\mathrm{FD} - f\mathrm{D} = 2\,\mathrm{CD}$,

$$\mathrm{F}f : \mathrm{FE} - \mathrm{E}f : : 2\,\mathrm{BC} : 2\,\mathrm{CD} \quad . \quad . \quad . \quad (2).$$

Wherefore we readily get from (1) and (2),

$$2\,\mathrm{CT} : 2\,\mathrm{CB} : : 2\,\mathrm{CB} : 2\,\mathrm{CD}, \text{ or } \mathrm{CT} : \mathrm{CB} :: \mathrm{CB} : \mathrm{CD};$$

that is, the semi-transverse CB is a mean proportional between CT and CD.

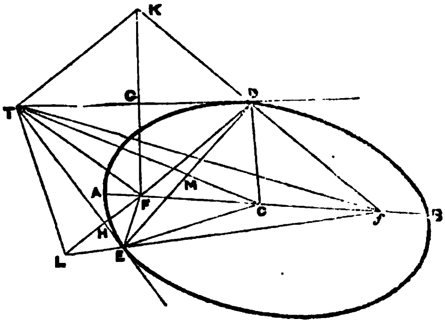
PROPOSITION VIII.

The straight line drawn from either focus of an ellipse to the intersection of two tangents to the curve makes equal angles with the straight lines drawn from the same focus to the points of contact.

Let TD, TE, tangents to an ellipse at the points D and E, intersect in T; draw to the focus F the lines DF, EF, TF; also draw FG, FH perpendicular to the tangents DT, ET, to meet the lines $f\mathrm{D}$, $f\mathrm{E}$, drawn from the other focus in K and L; and join TK, TL.

Then because the angle FDG = KDG (*Prop.* VI.), and the angles at G right angles, the triangles KGD, FGD, are equal, and therefore KD = DF. Wherefore $\mathrm{K}f = \mathrm{KD} + \mathrm{D}f = \mathrm{FD} + \mathrm{D}f = \mathrm{AB}$; and similarly, $\mathrm{L}f = \mathrm{AB}$. Whence $\mathrm{K}f = \mathrm{L}f$. The triangles FGT, KGT, are also evidently equal in all respects, and therefore KT =

TF. For the same reason, LT = TF; whence LT = TK. It will hence be obvious that the triangles TK*f* and TL*f* are equal in all respects, and therefore the angle D*f*T = L*f*T. Also if perpendiculars be drawn from *f* on the tangents produced, it may be shown in a similar way that the angle TFD = TFE. Consequently, the straight line FT drawn from the focus F makes equal angles with the lines FD, FE, drawn from F to the points of contact D and E; and also *f*T makes equal angles with *f*D, *f*E, the corresponding lines and angles with respect to the other focus *f*.

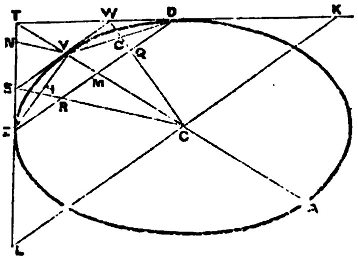


Cor. 1. It is shown in the proposition that TK = TL; and in a similar way it might be shown that perpendiculars from T on FD, FE, are equal; wherefore, *perpendiculars drawn from the intersection of two tangents to an ellipse on the lines drawn from either focus through the points of contact are equal.*

Cor. 2. Let the other lines be drawn as in the figure. Then because FH = HL, and FC = *f*C (C being the centre), the triangles TEF, TCF, ECF, are respectively equal to TEL, TC*f*, EC*f*. Wherefore the triangle TEC is one-half of the triangle TL*f*. For the same reason TCD is one-half of TK*f*. But it has been shown in the proposition, that TL*f* = TK*f*; therefore TEC = TDC. Whence the perpendiculars from D and E on TC are equal (the perpendiculars are not given in the fig.), and therefore DM = ME; that is, *the line drawn from the intersection of two tangents to an ellipse to the centre bisects the line which joins the points of contact.*

PROPOSITION IX.

Any diameter of an ellipse bisects all its own ordinates.



Let WS be a tangent to an ellipse at the point V; VA a diameter through the same point; and ED any ordinate parallel to WS (*Def.* 10); then the diameter VA bisects ED in the point M.

For let C be the centre of the ellipse, and W, S, the points in the tangent at V, in which tangents at D and E meet the tangent WS. Join SC, WC, cutting the chords DV, EV, in G, H, and the chord DE in Q and R. Draw also through the centre C the line LK parallel to DE, and from V, VN parallel to CS.

Now because the lines WS, DE, LK are parallel, the triangles VWG, SVH, are respectively similar to DGQ, REH; and since (*Prop.* VIII., *Cor.* 2), VG = GD, and VH = HE, those triangles are respectively equal in all respects. Hence VW = DQ, and SV = ER.

Again, by the similar triangles LCS, RES, CL : RE : : CS : SR.

Also by the similar triangles CWK, QWD, CW : WQ : : CK : QD.

But CS : SR : : CW : WQ. Hence CL : RE : : CK : QD.

Now, by *Prop.* VIII., *Cor.* 2, the line which passes through C, the centre, and the intersection of the tangents DK, EL (we do not know yet that this point of intersection is in VA), bisects the line DE, which joins the points of contact. Hence because LK is parallel to the line DE, LK is bisected in C, and therefore LC = CK. Wherefore by the last proportion, RE = QD. But it has been shown that RE = SV, and QD = VW; whence SV = VW. Consequently, RM = MQ, and therefore EM = MD; that is, the diameter VA bisects any of its ordinates ED in the point M.

Cor. 1. It was proved in *Prop.* VIII., *Cor.* 2, that C the centre, M the middle of ED, and the intersection of the tangents, were in the same straight line; and it has now been shown that C, M, V are in the same straight line. Wherefore *the tangents at the extremities of any chord of an ellipse meet in the diameter of that chord.*

Cor. 2. Since SH is parallel to NV (*Constr.*), and EH = HV, therefore ES = SN. But by the parallels NV, SC, and SV, EM, we have

CV : CT : : SN : ST, and SE : ST : : MV : VT;

and since ES = SN, we have

CV : CT : : MV : VT.

This gives, by alteration and division,

CM : CV : : CV : CT.

Hence, *if a tangent and ordinate be drawn to meet any diameter of an ellipse, the semi-diameter will be a mean proportional between the distances of the two intersections from the centre.*

This generalizes the property established in *Prop.* VII.

Cor. 3. *All the ordinates to the same diameter of an ellipse are parallel to one another.*

Cor. 4. *A straight line which bisects two parallel chords of an ellipse, and terminates in the curve, is a diameter.*

PROPOSITION X.

The square of any diameter has to the square of its conjugate the same ratio that the rectangle under the abscissas of any point has to the square of the corresponding ordinate.

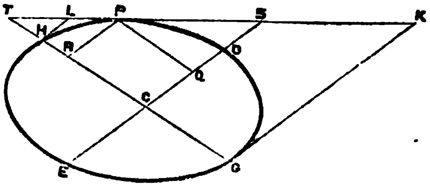
Let DE, GH be two conjugate diameters (*Def.* 10), and PQ an ordinate to the former, corresponding to the point P in the curve. Also let the tangent at P meet the conjugate diameters in S and T.

Then by *Cor.* 2 *of last Prop.*, and the property of duplicate ratio,

$$CQ : CD :: CD : CS, \text{ or } CQ^2 : CD^2 :: CQ : CS.$$

Hence, by division and similar triangles, remembering that $QP = RC$, we have

$$CD^2 - CQ^2 : CD^2 :: QS : CS :: RC : CT.$$



But by *Prop.* IX., *Cor.* 2, and *duplicate ratio*,

$$RC : CT :: RC^2 : CH^2; \text{ wherefore}$$
$$CD^2 - CQ^2 : CD^2 :: RC^2 : CH^2.$$

Now because C is the middle of ED, $CD^2 - CQ^2 = EQ \,.\, QD$; consequently $EQ \,.\, QD : QP^2 :: CD^2 : CH^2 :: ED^2 : HG^2$; that is, the square of the diameter CD is to the square of its conjugate HG as the rectangle under the abscissas EQ, QD, is to the square of the ordinate PQ.

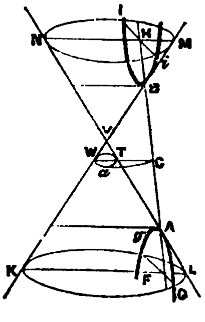
This is the general case of the property established in *Prop.* II.

COR. 1. *The rectangles of the abscissas of any diameter are as the squares of the corresponding ordinates.*

THE HYPERBOLA.

PROPOSITION I.

The squares of the ordinates of the transverse diameter are to one another as the rectangles of their abscissas.



As in the parabola and ellipse, let AVB be the transverse plane; AGIB the hyperbolic sectional plane which meets both sheets of the cone (*Def.* 3); AB the transverse axis which meets the two sheets of the cone in A and B; KGL, MIN, any two sections of the cone perpendicular to its axis. Also, let KL, MN, be the intersections of KGL, MIN, with the transverse plane, and GFg, IHi those with the sectional plane. Then it may be shown, as in the parabola (*Prop.* I.), that KL, MN are diameters of the circles KGL, MIN, and that the transverse diameter AB bisects Gg, Ii, perpendicularly in F and H. Hence by the similar triangles AFL, AHN, and KFB, MHB,

$$AF : AH :: FL : HN, \text{ and } FB : HB :: KF : MH.$$

Whence $AF \,.\, FB : AH \,.\, HB :: FL \,.\, KF : HN \,.\, HM$.

But by *Euc.* III. 35,

$$FL \,.\, KF = FG^2; \text{ and } HN \,.\, HM = HI^2.$$

Wherefore, $AF \,.\, FB : AH \,.\, HB :: FG^2 : HI^2$; that is, the squares of the ordinates FG, HI, are to one another as the rectangles of their corresponding abscissas.

The complete hyperbola, therefore, is situated on both sheets of the cone, and is composed of two separate and infinitely extended branches (*Def.* 1). Those branches, moreover, are divided by the transverse diameter into two parts, which are symmetrical with respect to that diameter.

PROPOSITION II.

The rectangle under the abscissas of any point in the hyperbola is to the square of the corresponding ordinate as the square of the transverse diameter is to the square of its conjugate.

(See the Fig. of last Prop.)

Let a circular section of the cone meet the middle of the diameter AB in C; and let C*a* be a tangent from C to this circle. Then C*a* is the conjugate axis to AB.

Let WT be the diameter of this circular section; then by the similar triangles AFL, ATC, and FBK, BWC, we have

$$AF : AC :: FL : TC, \text{ and } FB : BC :: FK : WC.$$

Hence, $AF \cdot FB : AC \cdot CB :: FL \cdot FK : TC \cdot WC.$

But since $AC = CB$, $AC \cdot CB = AC^2$, also (*Euc.* III. 35, 36) $FL \cdot FK = FG^2$ and $TC \cdot WC = Ca^2$. Wherefore

$$AC^2 : Ca^2 :: AF \cdot FB : FG^2,$$
$$\text{or } AB^2 : 4Ca^2 :: AF \cdot FB : FG^2;$$

that is, the square of the transverse diameter is to the square of its conjugate as the rectangle under the abscissas of the point G is to the square of the ordinate of the same point.

COR. Take p, a third proportional to AB and $2Ca$; then by the property of duplicate ratio, $AB : p :: AB^2 : 4Ca^2$. But by the proposition $AB^2 : 4Ca^2 :: AF \cdot FB : FG^2$. Wherefore, $AB : p :: AF \cdot FB : FG^2$. The quantity p which remains constant for different points in the curve is the *parameter* of the transverse diameter AB (*Def.* 13). Wherefore *the transverse diameter of the hyperbola is to its parameter as the rectangle under the abscissas of any point is to the square of the corresponding ordinate.*

PROPOSITION III.

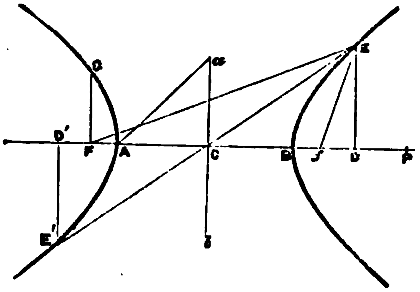
In the hyperbola, the square of the distance of either focus from the centre is equal to the sum of the squares of the semi-diameters.

Let AB be the transverse diameter of the hyperbola, *ab* its conjugate, F and *f* the foci, C the centre, and GF an ordinate at the focus F. Then by *Prop.* 2 and *Euc.* II. 6,

$$CA^2 : Ca^2 :: AF \cdot FB : GF^2 :: CF^2 - CA^2 : GF^2.$$

But F being the focus, $GF = \frac{1}{2}p$ (*Def.* 14), and therefore by *Def.* 13 $AC^2 : Ca^2 :: Ca^2 : \frac{1}{4}p^2$ or GF^2.

$$\text{Hence } CF^2 - CA^2 = Ca^2, \text{ or } CF^2 = CA^2 + Ca^2.$$



Cor. Join Aa; then, by *Euc.* I. 47, and the *Prop.*, $Aa^2 = AC^2 + Ca^2 = CF^2$; hence *the distance of either vertex of the conjugate diameter from either vertex of the transverse is equal to the focal distance of the centre.*

PROPOSITION IV.

If in the hyperbola a fourth proportional be taken to the semi-transverse diameter, the eccentricity, and the distance of an ordinate to any point in the curve (all estimated from the centre along the transverse axis), then the parts of the transverse axis intercepted between the extremity of the fourth proportional and the extremities of the transverse are respectively equal to the focal distances of that point.

(See the Fig. of last Prop.)

Let E be any point in the hyperbola, F and f the foci, and CP a fourth proportional to CB, Cf, and CD; then the segments AP, BP, of the transverse axis, intercepted between P and the extremities A and B of the transverse diameter, will be respectively equal to EF, Ef, the focal distances of the point E.

For by construction,

$CB^2 : Cf^2 :: CD^2 : CP^2$; hence since (*Prop.* III.) $Cf^2 - CB^2 = Ca^2$, $CB^2 : Ca^2 :: CD^2 : CP^2 - CD^2$. But (*Prop.* II.),

$$CB^2 : Ca^2 :: AD \,.\, DB : DE^2$$

$$\text{or } CB^2 : Ca^2 :: CD^2 - BC^2 : DE^2.$$

Wherefore, $CD^2 : CD^2 - BC^2 :: CP^2 - CD^2 : DE^2$.

Proceeding with this proportion as in the analogous case of the ellipse (*Prop.* IV.) we get

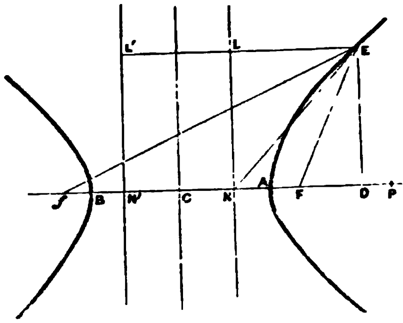
$$EF = AP, \text{ and } Ef = BP.$$

Cor. *The difference of the lines drawn from the foci to any point in the hyperbola is equal to the transverse diameter.*

PROPOSITION V.

The distance of the directrix from any point in the hyperbola has to the distance of the focus from the same point a constant ratio.

Let NL be the directrix with respect to the focus F, and N′L′ that with respect to the other focus f; then E being any point in the curve, the ratio of FE to EL, or of fE to EL′ is equal to the constant ratio of CF to CA (C being the centre).



For take in the transverse axis AB, CF : CA : : CA : CN, and draw NL perpendicular to the axis; then NL is the directrix with respect to the focus F (*Def.* 15). Also, as in the *last Prop.*, take CP a fourth proportional to CA, CF, and CD.

Then we have

CF : CA : : CA : CN, and CF : CA : : CP : CD.

Hence

CA : CN : : CP : CD, or CA : CP : : CN : CD.

Wherefore, by division,

AP : CA : : DN : CN; or since AP = EF (*Prop.* IV.),
and DN = EL, EF : EL : : CA : CN : : CF : CA.

And if N′L′ be the directrix with respect to the other focus f, it may be shown in a similar way, that

Ef : EL′ : : CF : CA.

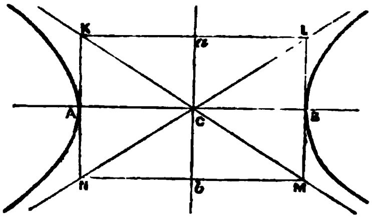
PROPOSITION VI.

The straight line which bisects the angle adjacent to that which is contained by two straight lines drawn from any point in the hyperbola to the foci, is a tangent to the curve at that point.

The demonstration of this, which is exactly similar to the analogous case for the ellipse (*Prop.* VI.), is left for the student's exercise.

The Asymptotes.

Def. 16. If through A, one of the vertices of the transverse diameter of the hyperbola, a straight line KAN be drawn, equal and parallel to $a\,b$ the conjugate diameter, and bisected in A by the transverse diameter; the straight lines CK, CN drawn through the centre, and the extremities of that parallel, are called *asymptotes* to the hyperbola. The asymptotes also of two opposite hyperbolas are common to both.



There are many remarkable properties connected with the asymptotes to the hyperbola; but the limits of this course do not admit of our entering upon them. It may be stated, however, *that the hyperbola and its asymptotes, when produced, continually approach to each other, but meet only at an infinite distance.* (*See* Vol. I., p. 306).

CONJUGATE HYPERBOLAS.

DEF. 17. If *ab* be taken for the transverse diameter, and AB for its conjugate, of other two hyperbolas in the spaces KCL, MCN (*a* and *b* being the vertices of those diameters); then those two other hyperbolas are said to be *conjugate to the former.* When all four are mentioned, they are called *conjugate hyperbolas.*

Additional Problems and Theorems for Exercises.

1. Prove that if any two diameters of a parabola be produced to meet a tangent to the curve; the segments of the diameters between their vertices and the tangent are to one another as the squares of the segments of the tangent intercepted between each diameter and the point of contact.

2. A parabola is given in position, to find its directrix and focus.

3. If any chord be drawn parallel to the tangent of a parabola, to meet the curve in two points; and if from its extremities and from the point of contact ordinates be drawn to the axis, then double the ordinate of the point of contact will be equal to the sum or difference of the other two, according as they are situated on the same or different sides of the axis.

4. If from the point of contact C of a tangent to a parabola, any chord CL be drawn, and another line parallel to the axis to meet the chord, curve, and tangent, in K, E, I; then IE : EK : : CK : KL.

5. If from the point of contact of a tangent to a parabola, two lines be drawn to the vertices of any two diameters, each to intersect the other diameter; then the line joining the two points of intersection will be parallel to the tangent.

6. If the chord of contact of two tangents to a parabola pass through the focus, the tangents will intersect at right angles to one another in the directrix.

7. Let TP, *tp*, be tangents to a parabola at the points P and *p*, and PM, *pm*, ordinates on the axis from the same points; then

$$(1)\ \tan TPM : \tan tpm :: PM : pm;$$
$$(2)\ \tan TPM - \tan tpm = 2 \tan p\,PM.$$

8. Show how to draw a tangent and normal to a given parabola from a given point without the curve.

9. Prove that any chord of an ellipse drawn through the focus is so divided in that point that four times the rectangle contained by its segments is equal to the rectangle contained by the chord and the parameter of the transverse diameter.

10. Let a tangent to an ellipse intercept the four perpendiculars on the axis, from the centre, the two extremities of the axis, and the point of contact; then will those intercepted perpendiculars be proportionals.

11. Prove that if a right cylinder be cut by a plane obliquely the section will be an ellipse.

12. Show how to draw a tangent to a given ellipse from a given point without the curve.

13. Prove that chords drawn through the foci of an ellipse or hyperbola are proportional to the squares of their parallel diameters.

14. Prove that the segment of the normal, intercepted by the principal diameter of the ellipse or hyperbola, is equal to the distance of the focus from the centre.

PROJECTION.

DEFINITIONS AND FIRST PRINCIPLES.

1. (*a*) If lines be drawn from every point of an object on the opposite side of a pane of glass to the eye (representing the rays of light passing from the points of the object), and if the points in which these lines cut the pane be marked upon it, we have the *picture* or *projection* of the object. This projection is the *perspective* of the object.

(*b*) If any opaque figure be interposed between a light and a screen, the shadow of the figure upon the screen is the *projection* of the figure. This is the *radial projection* of the object.

(*c*) Although there is a physical difference in the circumstances of these two projections, yet geometrically, the difference is so slight and subordinate as to not justify a separate general discussion or explanation of the two cases. They are therefore usually considered as one.

(*d*) When the *projecting point* (the eye in the first case and the luminous body in the second) is brought nearer to the body, the projection is enlarged; and when taken more distant, the projection is diminished. For as the point approaches the body, the bounding rays become angularly more and more expanded; and as the point is removed from the body, they approach nearer and nearer to parallelism. Each case, however, is subject to limitations which will be pointed out hereafter. The general fact is sufficient for our present purpose.

(*e*) The general idea of projection as a branch of geometry has been obtained from the considerations above, by ideally generalizing the objects—by abstracting all considerations of the materiality of light, the figure which by interrupting the light causes the shadow, and of the surface upon which the shadow is cast; and thus retaining only the purely geometrical idea of *direction in the rays*, of the *defining conditions* of the intercepting or projected figure, and likewise *those* of the surface upon which it is projected.

2. The only two cases in which the idea thus generalised is applicable directly and immediately, as far as regards the *positions* of the rays, are:—

(*a*) That they shall be parallel to a given line;
(*b*) That they shall all emanate from a given point.

3. The figure to be projected may either be:—a point, a straight line, a curve line, a definitely bounded plane figure, or a curve surface. This is called simply the *figure*.

4. The surface upon which the shadow is hypothetically cast, is called the *figure* (plane, sphere, cone, etc.) *of projection*. Most frequently and for the most useful applications of the method of projection, this is a plane, and called the *plane of projection*. Sometimes, however, projections are made upon spherical, conical, or cylindrical surfaces.

5. The hypothetical shadow itself is called the *projection* of the figure; and the lines which are drawn from each point of the figure to the corresponding one of its projection, are called the *projectors*.

6. When the projectors emanate from, or converge towards the same point, the method is called *polar projection*, and comprehends both the cases supposed in (1*a* and 1*b*); when they are parallel, it is called *parallel projection;* and when they are perpendicular to the plane of projection, it constitutes the *orthographic projection*.*

7. The point in which any given line cuts the plane of projection is called the *trace* of that line; and the line in which any given plane cuts the plane of projection is called the *trace* of that plane. The same term is also applied to the intersections of any given curves or surfaces with the plane of projection.

It is manifest that the trace of every line in a given plane upon the plane of projection is situated in the trace of that plane upon the plane of projection.

Moreover, when the traces of two lines which are known to be situated in a plane are given, the trace of that plane is given; and it is the indefinite line drawn in the plane of projection through these two traces.

8. Any geometical figure is said to be *given* when it is fixed by adequate conditions, by conditions which do not allow it to undergo any change without a previous change in one or more of those conditions.

9. By the term *inclination*, applied to lines or planes, is meant the ordinary geometrical idea attached to them by writers on planes and solids: but it is to be kept in view, that we shall here always consider the inclination to be from one specified side of the point about which a line may revolve or one specified side of the line about which the plane revolves, to form its inclination with the plane of projection. Thus:—

FIG 1.

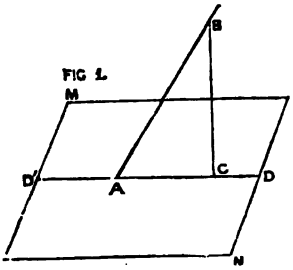
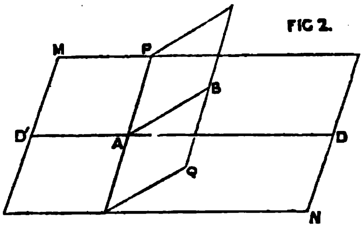


FIG 2.



If, in Fig. 1, D'AD be the line from which the inclination of AB to the plane MN is to be estimated, it is to be predetermined whether it shall be with the branch AD or AD′: and, similarly, if, in Fig. 2, AP be the trace of the plane PQ upon MN, it must be predetermined whether the inclination is to be estimated from the portion of the plane to the right of AP or from that to the left—in other words, whether AB is to be compared with AD or AD′ in defining the inclination of PQ to MN. In speaking of the inclination as "given," this is always, then, understood as an element in the conditions by which it is given, whether so expressed or not.

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* This, and the case of the polar projection, which relates to perspective, are the only ones that were studied in this country till a very recent period.

10. The simplest form in reference to one plane of projection in which a plane can be given is by means of its trace and inclination; and in this way, when nothing to the contrary is expressed, it is to be understood as given in the following pages.

It may, however, be otherwise given; as by means of two lines in it; of a point and line in it; of three points in it; of being drawn through a given point parallel to a given plane; or through a given line parallel to another given line.

11. A line is given in the following ways:—

(*a*) By its position in a given plane in reference to the trace of that plane;

(*b*) By its trace, inclination, and the line in the plane of projection to which its inclination is referred. This line, it will be presently seen, is the projection of the given line.

(*c*) By giving two points through which it passes, either in the ways designated by (*a*) or (*b*), or by any others.

Many other methods of exhibiting the data of a line occur in practice, of which examples will hereafter occur; but for use, they are almost invariably reduced to one or other of these.

12. A point is most simply given by means of its projection, and the magnitude and direction of its projector. The direction of the projector is usually given by means of a pattern-projector; or, in other words, by giving one line to which the projectors are all parallel, or a point through which they all pass. In the *orthographic projection*, this line being always perpendicular to the plane of projection, need not be further specified; and when the projector is said to be given, its magnitude alone is meant, as the projection itself being in this case given, the direction of the projector becomes fixed without further intimation.

A point, however, is often most conveniently given in practice, by giving its position on a given plane in reference to the trace of that plane on the plane of projection. It occurs in this shape in almost every class of actual problems.

Often, indeed, it is given by both methods simultaneously; viz., by its projection and projector in reference to a plane which is itself given by means of its trace and inclination to the plane of projection. In this way almost all polyhedral solids are given, the plane in question to which the angular points are referred being one of the faces of the figure, and the trace of that plane on the plane of projection serving as the common trace for reference to the entire system.

A point, however, may be (and often is) given as the intersection of two given lines; as the intersection of three given planes; and in other ways; but for the most part these require subsidiary constructions to reduce the data to the simpler forms above enunciated.

13. All plane figures whatever (as angles, polygons, circles, or

planė curves) are given by giving them as a line was given, by stating them in reference to the trace of that plane upon the plane of projection, and giving that plane by means of its trace and inclination.

14. In every case of the projection of a point upon a plane, the projection is a single point.

For through that point only one line can be drawn either parallel to a given line, or to emanate from a given fixed point; and this line can cut the plane of projection in only one point: that is, the projection of a point is a single point.

15. In every case of the projection of a straight line upon a plane, the projection is a single straight line.

For, since all the projectors are parallel to one another, or they all converge to a given point, they are all in one plane (the *projecting plane*) with the given line; and the trace of this plane on the plane of projection will (*Art.* 7) contain the projections of all the points of the given line. Moreover, this projecting plane can only cut the plane of projection in one line; and hence the projection of the given line is a single line.

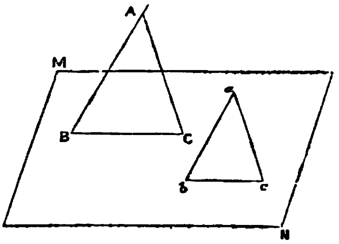
16. In every case, if a given line be parallel to the plane of projection, its projection will be parallel to itself; and if the given line meet the plane of projection, its projection will always pass through its trace upon that plane.

For, in the first place, the line is projected by a plane, upon a plane parallel to that line, and hence (*Pls. and Sols.*, *Chap.* I., *Prop.* II.) the projection of the line itself is parallel to the original.

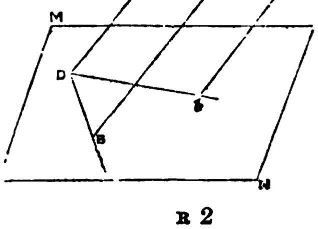
And in the second, the trace of the line being in the plane of projection, the projector of that point also meets the plane of projection in the same point that the line itself does; that is, the trace is one of the points in the projection of the line.

17. The *parallel projections* of parallel lines are parallel; and their *polar projections* meet in a point.

For, let MN be the plane of projection, and AB, *ab* two parallel lines which are to be projected by lines parallel to a given line: then, if two lines AC, *ac* be drawn parallel to the projecting pattern (12) to cut MN in C and *c*, the lines BC, *bc* will be the projections of AB, *ab*. But, since AB, AC are parallel to *ba*, *ac*, the projecting planes ABC, *abc* are parallel; and hence their sections BC, *bc* with MN are parallel.



Again, let C be the projecting point, and AB, *ab* two parallel lines: draw CD to meet MN in D parallel to AB, *ab*. Then the projecting planes for AB, *ab* pass through CD, and hence through D, in which they intersect MN. Whence also the polar projections of AB, *ab* pass through D.



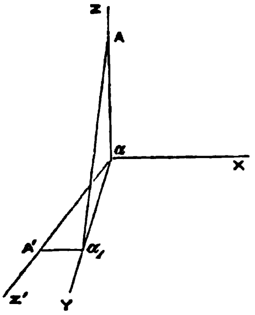
18. The *traces* of parallel planes are parallel; which is only another enunciation of *Prop.* I., *Chap.* I., *Planes and Solids*.

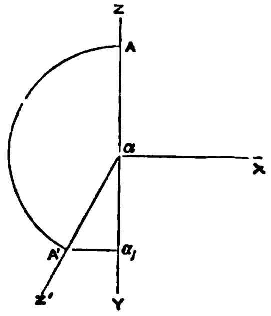
PROPOSITION I.

ORTHOGRAPHIC PROJECTION.

Given a point in a given figure plane to construct its projection, and the magnitude of its projector.

Let ZaX (first Fig.) be the figure plane, and YaX the plane of projection; and let A be the given point in it, whose projection on XaY is required to be found by a construction *in a plane*.





Now, the point A being given in the given plane XaZ, the line Aa perpendicular to its trace aX is given in magnitude and position. From a draw aY in the plane XaY perpendicular to aX. Then aA and aY being perpendicular to aX, the plane AaY is perpendicular to aX, and therefore, to each of the planes XaY, XaZ passing through it (*Pls.* II. 18); and YaZ is the inclination, which, by the problem, is given. Lastly, since A is in the plane YaZ, and this plane is perpendicular to the plane of projection, the projector $Aa_{,}$ of the point A is in this plane; and since the hypothenuse Aa of the right-angled triangle $Aaa_{,}$, and the angle $Aaa_{,}$ are given, the sides $aa_{,}$ and $a_{,}A$ are also given.

Let us now turn the plane XaZ about aX, and the plane YaZ about aY till they coalesce with the plane of projection: then aZ will coincide with aY (either actually if brought forward, or in extension if revolved backwards), and $A'a_{,}$, that is $Aa_{,}$ will still be perpendicular to aY, and the perpendicular $A'a_{,}$ will pass through the projection of A, and be equal to the projector $Aa_{,}$.

By a little consideration of the indications of the figure in its present state, we shall see that the proposed conditions will enable us to construct it by means of operations performed wholly in one plane.

For, the point A being given with respect to the trace, the point A and perpendicular Aa are given, and can be laid down on the plan. Produce this perpendicular to Y (Fig. 2): then the projection of A is in aY. Also the inclination YaZ' is given, and the line $aA'Z'$ being drawn to make that angle with aY, and $A'a$ being set off equal to aA, and finally, the perpendicular $A'a_{,}$ being drawn to aY, we obtain the point $a_{,}$—which is the required projection of A viewed as in its original position.

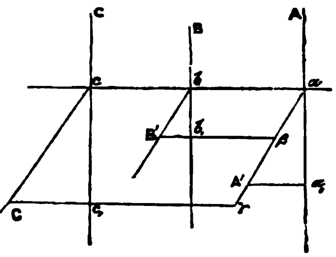
SCHOLIUM 1. It will be readily seen from the preceding proposition, that the only use that has been made of the first figure was to enable us to *investigate the properties of the real figure*, so as to deduce relations which should enable us to dispense with its actual use in the construction itself. The student is urged to keep this distinctly in his mind as the essential spirit of all projective methods, which have a *practical object* in view; and he will find that throughout this entire work the same fundamental characteristic is readily distinguishable. The figure which represents the lines, etc., in space is used for reasoning only—never used or implied in construction.

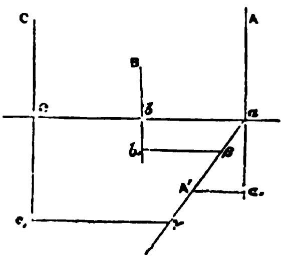
This distinction has not been brought prominently forward in the previous branches of geometry. In plane geometry, the data, the construction of the problem, and the subsidiary lines employed in analysis and demonstration, are all in one plane; and hence there is no necessity for paying the least attention to the circumstances under which those lines were supposed to have been introduced, except as far as their related properties are concerned. In the consideration of properties of lines, planes, or other figures in space, the properties were absolutely deduced and the hypothetical constructions were performed, without the restriction to special means; and hence the system required no such restrictions in the manner of performing the assigned operations.

SCHOLIUM 2.—In the case which we have exemplified, the point A is *above* the plane of projection, and the figure plane is turned to the *opposite side of the trace* from that on which the projection is situated. Either of these circumstances might have been reversed, or even both of them together.

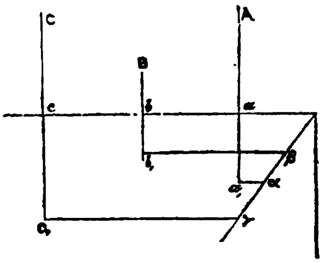
The figures ordinarily given in works on projection suppose the figure plane to be turned *upon* the region that contains the projection: but the mixture of the lines which are involved in delineating a figure and its projection, render the separation generally more convenient in complex cases. We shall, however, sometimes employ one and sometimes the other mode, for the sake of rendering every combination familiar to the mind and eye of the student.

SCHOLIUM 3.—When several points in the figure plane are to be orthographically projected, the same process which has been applied to one may be, obviously, applied to each of the others in succession: but in practice, the work may be in a slight degree abridged, and the number of lines to be actually drawn, somewhat lessened.





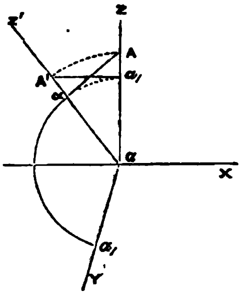
Let, for instance, A, B, C, be projected, each according to the method and independently of each other, as in the first figure: and produce $B'b_,$, $C'c_,$ to meet aA' in β and γ: then $b\beta$ and $c\gamma$ are parallelograms, and $a\beta = bB' = bB$, and $a\gamma = cC' = cC$. Whence, if we take upon *one* of the lines making the given inclination with the profile trace, as aA' (Fig. 2), and set off upon it the distances $a\beta$, $a\gamma$, aA' equal to



Bb, Cc, Aa, and draw parallels $\beta b_,$, $\gamma c_,$, A'a to the trace, meeting Bb, Cc, Aa (produced if necessary) in $b_,$, $c_,$, $a_,$, these points will be the projections required.

It is very common, for preserving the uniformity of the figure to take instead of a line through one of the points a, b, c, concerned in the problem, one altogether unconnected with the inquiry, as in Fig. 3; the import of which will be at once apparent without verbal explanation.

Scholium 4.—A slight modification of the primary construction is sometimes employed; and it will appear obvious from the following transformation of the figure.



Suppose the profile plane $A\alpha a_,$ of the system (Fig. 1 of the *Prop.*) to be turned about Aa till it coincides with XaZ, and then the compound plane figure A'aX to be made to coincide with the plane of projection. Then the projector will take the position Aα perpendicular to aA', as in the annexed diagram; and aα will be the distance of the projection $a_,$ from the trace aX. Setting off this distance upon Aa, or upon aY, will give the projection sought, on the upper or lower side of the trace aX respectively.

That this gives the same point as the former construction is thus made evident. Make a A' = aA, and join $A'a_,$. Then the triangles $A\alpha a$, $A'a_,a$ have two sides equal each to each, $A'a = aA$, $a\alpha = \alpha a_,$, and the angle at a common: and hence, the bases are equal, and the remaining angles are equal, each to each. Whence $A'a_,a = A\alpha a$; and this last is a right angle. The point $a_,$ is, therefore, the same as would have been found by the first process.

Also Aα is the orthographic projector of the point A, as is obvious.

Scholium 5.—The converse problem is sometimes required in conducting a course of consecutive and dependent constructions; and likewise in projecting upon "*oblique sites*," which is often employed in military works. It is,—

Given the projection of a point situated in a given figure plane, to fin the point itself.

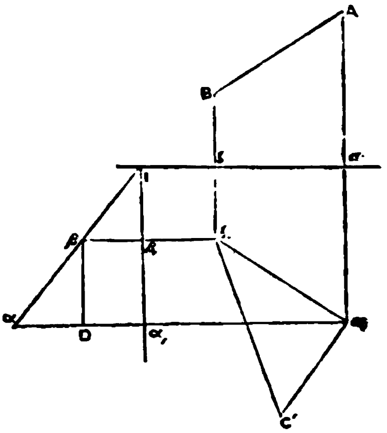
In this case $a_,$ is given (Fig. 2 of the *Prop.*); construct aZ' as before, and draw $a_,A'$ perpendicular to aY: then aA' being set off above or below a upon ZY, gives A the original point.

PROPOSITION II.

Given the trace and inclination of the figure plane, and the position of a given line in that plane, to project orthographically the line, and to construct the inclination of the line to the plane of projection.

(a.) It is obvious that, if by the preceding proposition we project two points of the given line, then the line through these will be the projection of the given line. Whence follows a *first construction*.

Make Iβ = bB, and Iα = aA, the angle at I being the inclination as in the former problem; and find $b_{,}$, $a_{,}$ the projections of B, A; and the line $a_{,}b_{,}$ is the projection sought.

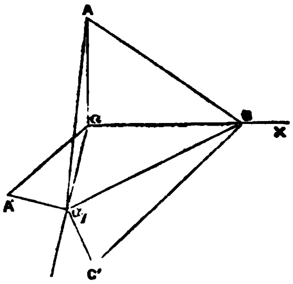


To find the inclination of the line AB to the plane of projection, if we draw in the projecting plane of AB the line BC parallel to the plane of projection, this will form with AB the same angle that $a_{,}b_{,}$ forms with it, and consequently that AB forms with the plane of projection. Now, in this right-angled triangle we have AB the given line, BC the equal of $b_{,}a_{,}$ which is already found, and CA the difference of the orthographic projectors of A and B which are exhibited as $\beta\beta_{,}$, $\alpha\alpha_{,}$ in the figure. Whence, if we make $a_{,}$C′ perpendicular to $a_{,}b_{,}$ and either make $a_{,}$C, equal to αD the difference of $\alpha\alpha_{,}$ and $\beta\beta_{,}$, or make $b_{,}$C′ equal to AB, we shall have a triangle equal in all respects to that just verbally described, and $a_{,}b_{,}$C′ is the inclination of AB to the plane of projection.

This method, though capable of application in all possible cases, is often superseded in practice by more facile processes, of which two are given here.

(*b*.) When the figure line can be produced to meet the trace of the figure plane within the limits of the drawing, the following method is that most frequently employed in practice.

Let AB be the figure-line, produced to meet the trace in B: find, as before, the projection $a_{,}$, of one point A, and join $a_{,}$B. This is the projection sought.



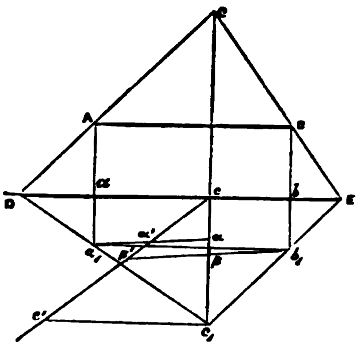
For $a_{,}$ is one point by construction; and B is a point also in the projection by *Art.* 16.

Again, make $a_{,}$C′ at right angles to $a_{,}$B, and equal to $a_{,}$A′, and join BC′: the angle $a_{,}$BC′ is the inclination of AB to the plane of projection.

For it is obvious that $a_{,}$C′B is but the triangle $a_{,}$AB turned about $a_{,}$B on the plane of projection, and hence that the angles $a_{,}$BC′ and $a_{,}$BA are equal, whilst the latter of these is the inclination itself.

(*c*.) The following method also possesses advantages in practice which will often be recognised.

Let A, B be points in the line, and take any other point C in the figure plane that may appear convenient, and draw CA, CB to meet the trace in D, E: find the projection $c_{,}$ of C; draw D$c_{,}$,

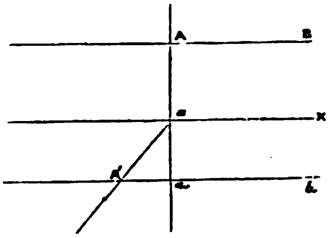


$Ec_,$ which will be the projections of CD, CE; and draw $Aaa_,$, $Bbb_,$ parallel to Cc, meeting $Dc_,$, $Ec_,$ in $a_,$ and $b_,$. Then $a_,b_,$ is the orthographic projection of AB.

For, always after the figure plane is turned round on the plane of projection, the point and its projection are in one line perpendicular to the trace; as $Aa_,$ is perpendicular to DE. Also, the projection of A is in $Dc_,$, and hence $a_,$ the intersection of Aa and $Dc_,$, is the projection of A. In like manner for $b_,$.

The inclination is found as in the first construction, the altitudes or projectors being $\alpha\alpha'$, $\beta\beta'$ found by drawing parallels from $a_,$ and $b_,$ to the trace.

(*d.*) When the given line is parallel to the trace, it will be parallel to the plane of projection. It will in that case be sufficient to construct the projection of one point A, and draw through $a_,$, a parallel $a_,b_,$ to the trace aX.

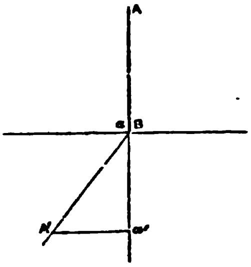


The inclination is zero; and since for all the points the projectors are of the same length, the same result is obtained by construction as by our previous reasoning we were led to deduce.

(*e.*) If the given line AB be perpendicular to the trace, only one point will be necessary to be found.

For find the projector $a_,$ of A in it as before; and then a itself will be the projection of the trace of the line.

The inclination of the line AB is in this case the same as the inclination of the figure-plane itself; and is therefore given.



(*f.*) If the figure-plane be perpendicular to the plane of projection, the projections of every line in it will be upon the trace.

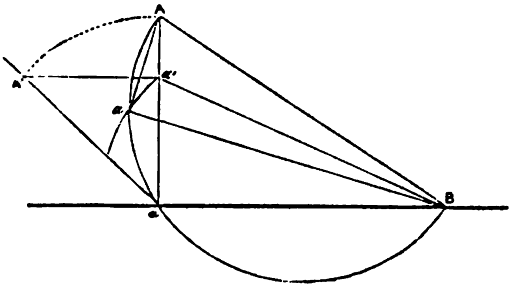
The constructions will terminate in the trace, and the inclinations of the lines to the plane will be the same as their inclinations to the trace.

Scholium. The converse case often occurs in a series of consecutive constructions, viz.:—

Given the projection of a line with the trace and inclination of the plane, to find the original line.

The requisite operations follow so readily from the construction of the direct problem, as to scarcely call for special remark: yet we shall give one example on account of its frequency of use.

Let aB be the trace of the given plane, and AaA' its inclination; and let a'B be the projection of the line in the given plane. On AB

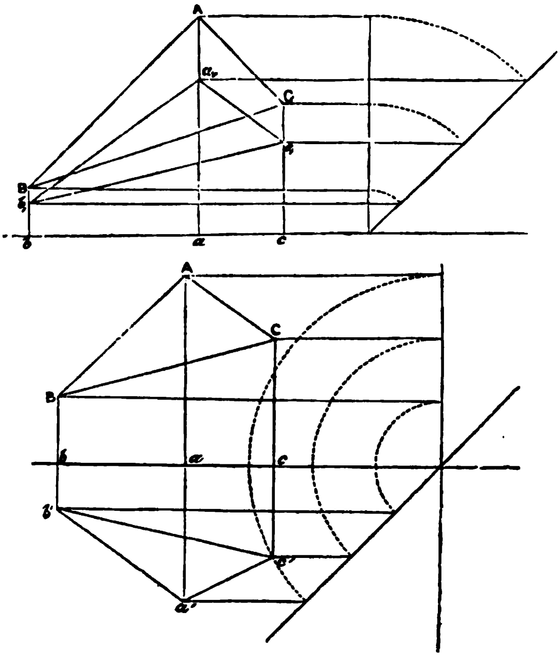


describe a semicircle and join Aa; it will pass through a' the projection of A, and a'B will be the *length* of the projection of AB. Make, therefore, Bα = Ba' and draw Aα. Then AαB is the projecting triangle, and αBA the angle of inclination of the line.

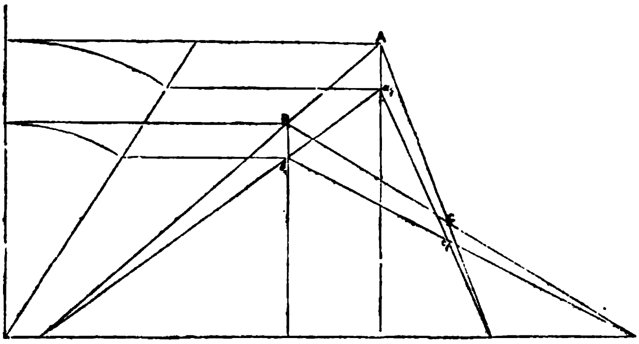
PROPOSITION III.

Given any rectilineal figure in a given figure plane to construct its projection.

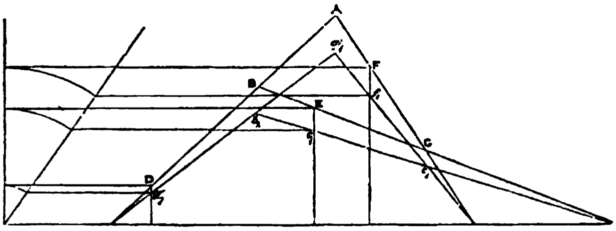
(*a.*) We may project the *angular points* (as in *Prop.* I.) and draw lines to join them. Thus in the triangle ABC we should have the figure as follows, according as we keep the figure after revolution, and the projection in the same or different sides of the trace of the figure plane.



(*b*.) We may project the *sides* of the given figure, viewed as indefinitely extended lines (as in *Prop.* II.): these projections by their mutual intersections give the limited projection of the figure. To render this method useful, and avoid a great deal of unnecessary labour, and the consequent complexity of the figure, some degree of discretion is necessary. Take, for example, a triangle ABC, and let the sides AB, BC, CA be projected by taking points D, E, F *arbitrarily* in these respective sides, as those which with the traces are to define the positions of the sides of ABC. Then the figure would resemble the following:—

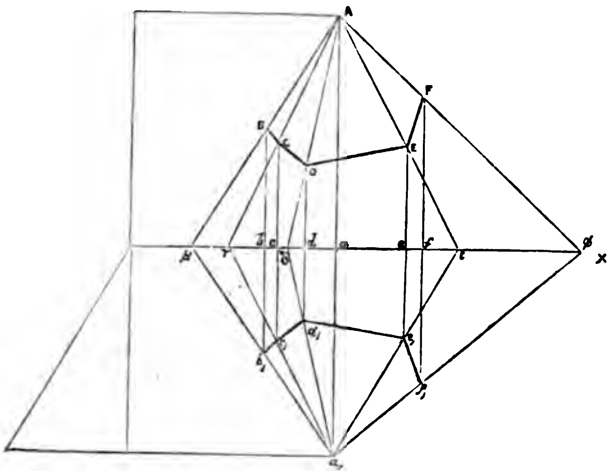


The complexity of such a figure is sufficiently discouraging in practical operations; and evidently, this complexity becomes greater with every new line of the figure to be projected. This complexity is, however, in a great degree evaded by taking, instead of D, E, F, some specific points either in the figure, or having known relations of position with respect to it. It is impossible to give any general rule for this choice: but the nearest approach to generality is to select an intersection of a line with an adjacent one—in fact an angular point of the figure—as the constructive one. The same triangle so constructed in projection would have its delineation as follows: but, on the whole, the method itself is not the most facile, and is only dwelt upon here for the sake of completeness of discussion.



(*c*.) In a practical point of view, the greatest facility is afforded in construction by the following process. As the method is precisely the same how numerous soever the lines may be, we shall give it in detail for the irregular hexagon ABCDEF.

Produce some two adjacent edges AB, AF to meet the trace *a*X of



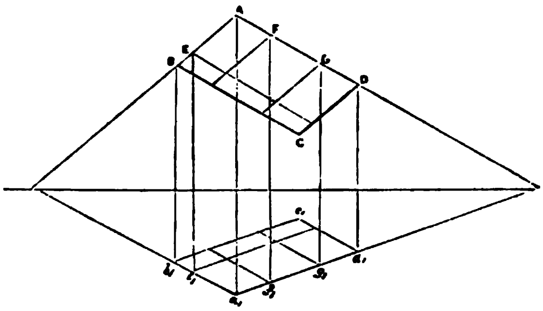
the figure plane in β, ϕ, and project the point A in $a_{,}$ by *Prop.* I.; draw AC, AD, AE to meet the trace in γ, δ, ϵ, and then draw $\beta a_{,}$, $\gamma a_{,}$, $\delta a_{,}$, $\epsilon a_{,}$, $\phi a_{,}$. These will be the projections of Aβ, Aγ, . . . Aϕ; and hence the projections of B, C, . . . F in them will be in $\beta a_{,}$, $\gamma a_{,}$, . . . $\phi a_{,}$. But these projections are also in lines B*b*, C*c*, . . . F*f* perpendicular to the trace; and hence at their respective intersections $b_{,}$, $c_{,}$, $d_{,}$, $e_{,}$, $f_{,}$. We are, therefore, only required to draw these two sets of lines $\beta a_{,}$ and B*b*, . . . to obtain the angular points of the projection; and these being joined in the same order as the original points, the projection is complete.

(*d.*) It will sometimes happen that no two adjacent sides are well adapted for the application of this method. In such case any arbitrary but more convenient point may be assumed as that to be projected; and the other points found by drawing lines through the given angular points to the trace, as was done in the preceding construction. Very often such a point may be conveniently found by producing two alternate sides, or indeed any two sides of the figure to meet; but in all cases the selected point should be such as to throw all the lines from it to the given angular points in such positions as to cut the trace within the limits of the drawing.

For example, if in the preceding figure the pentagon BCDEF had been given, no *two adjacent sides* of it would have been well fitted for the operation; but by taking the point A arbitrarily as in the figure, we should have found $b_{,} c_{,} d_{,} e_{,} f_{,}$, as already shown with much facility.

(*e.*) It will sometimes happen that where one or more pairs or sets of parallels enter into the composition of the figure, the work may be still further simplified by the application of the principle that *the projections of parallel lines are parallel.*

Let there be given, for instance, the parallelogram ABCD, having a line drawn through E parallel to AD or BC, and lines from F and G parallel to AB or CD; then of two adjacent sides AB, AD, as before, find the projections of A, E, B, F, G, D; and draw parallels from $e_{,}$, $b_{,}$, $f_{,}$, $g_{,}$, $d_{,}$: these will constitute the projection required.

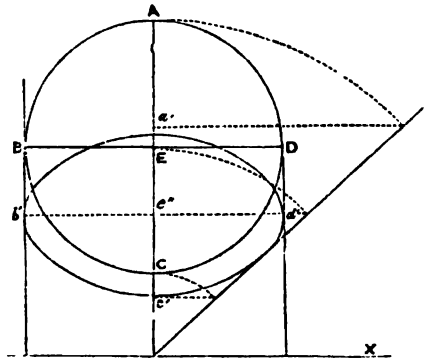


Scholium. Having given the trace and inclination of the figure plane, we can always by the reversal of the processes find the original figure which corresponds to any given projection.

PROPOSITION IV.

A circle is given in a given plane to construct its projection.

(*a.*) Draw diameters AC, BD perpendicular and parallel to the



trace: these will be projected into *a'c'*, *b'd'*, the minor and major principal diameters of the ellipse which represents the circle in projection. The ellipse may then be traced by any of the methods usually given for that purpose.

(*b.*) If the curve is to be traced by hand, it will be better to take eight or twelve points, or if large and very great accuracy is required, sixteen; and find the projections of these, as so many true points in the projection.

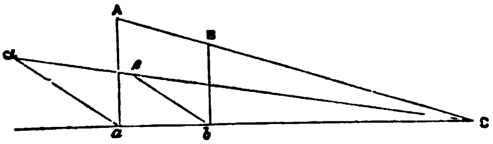
Scholium. This simple problem is given apart from the more general problems on the circle and conic sections, solely on account of its frequent use in practice. The method (*b*) will obviously apply to any figure whatever; and it is, in fact, the process commonly used by draughtsmen in all cases that occur of curve lines, including the circle.

PROPOSITION V.

(1.) *Given the projections and projectors of two points to find the trace and inclination of the line through them:*

(2.) *Given the projections and projectors of three points to find the trace and inclination of the plane through them.*

(1.) Let Aa, Bb be the two projectors, and AB be the line itself:

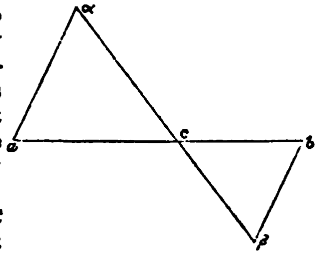


then the plane through Aa, Bb, will be the projecting plane of AB, and AB, ab will meet in C the trace of AB. But we have here given the points a, b, and the magnitudes of Aa, Bb at right angles to the plane, and therefore to the line ab. If now we turn the plane A$ab$B about ab on the plane of projection, Aa, Bb will still be perpendicular to a, b, as in the diagram. Whence we have this plane construction:—

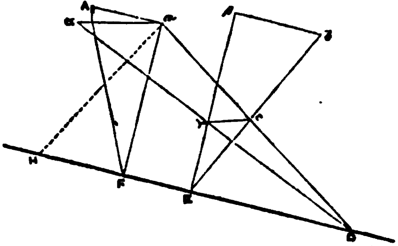
Join ab, and draw aA, bB at right angles to it, and equal to the given projectors: then AB, ab meet in C the required trace; and C is the inclination of the line to the plane of projection.

We may remark, that the point C would have been found by drawing the projectors $a\alpha$ and $b\beta$ parallel to each other in any direction soever, so as to make an angle with ab, precisely as when their positions were perpendicular to ab. When the *trace only* is required, this method is effective, but it does not give the inclination of the line without a separate and supplementary process.

Also, if the points be on *different sides* of the plane, the parallels must be taken in *opposite directions* to find C.



(2.) Let a, b, c be the given projections of the three points A, B, C (Fig. 3); draw the projections of any two of them ac, bc, and in any direction parallel to each other draw $a\alpha$, $b\beta$, $c\gamma$, equal to the three given projectors: then $\alpha\gamma$, ac meet in one point D, and bc, $\beta\gamma$ in another point E, which (by the preceding) are the traces of AC, BC. The line DE is the trace of the plane through them, and the trace of AB will also be in the same line. Whence we have found the *trace* of the plane ABC.



Again, from any one of the given projections, as a, draw aF perpendicular to the trace DE, and aA′ parallel to DE and equal to the projector $a\alpha$, and join A′F: then A′Fa is the inclination of the plane ABC.

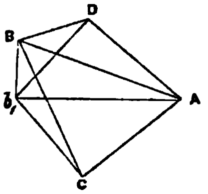
For, evidently, aFA′ is the triangle aFA, when turned on the plane of projection, of which the angle at F is the inclination of the plane to the plane of projection.

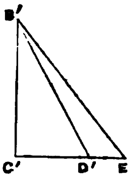
It may also be remarked, that the construction is a little abbreviated by making FH = $a\alpha$, and drawing aH; for the triangles aA′F and HFa are in all respects equal, and therefore the angles A′Fa, HaF are equal.

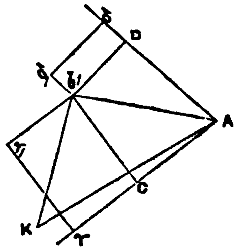
PROPOSITION VI.

Given the traces and inclinations of two planes, to find the projection and inclination of their line of intersection.

If we take any point B in the line AB (first Fig.) of intersection of the two planes, and draw the perpendicular B$b_{,}$ on the plane of projection: then, if $b_{,}$A be drawn to the intersection A of the traces AC, AD of the two planes, $b_{,}$A will be the projection required; and the result will be the same wherever in the intersection AB the point B be taken.



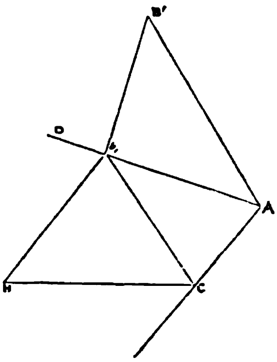




Conceive that through B$b_{,}$ planes B$b_{,}$C, B$b_{,}$D be drawn perpendicular to AC, AD: then BC$b_{,}$, BD$b_{,}$ are the inclinations, and therefore given by the problem. Whence, for any assumed value of B$b_{,}$, the *magnitudes* of $b_{,}$C, $b_{,}$D perpendicular to AC, AD are given. Whence $b_{,}$ is in the intersection of two lines parallel to AC, AD, and whose distances from those lines, for any assumed value of B$b_{,}$ are given. Whence we have the following construction:—

Let C′ (second Fig.) be a right angle taken in any convenient position on the paper, and B′ any point in B′C′: make the angles C′D′B′, C′E′B′ equal to the given inclinations of the planes, thus giving C′D′, C′E′: at these distances ($\delta\delta_{,}$ and $\gamma\gamma_{,}$) from the traces draw parallels (third Fig.) to the traces, meeting in $b_{,}$: then $b_{,}$A is the projection of the line AB in which the planes intersect.

To find the inclination of the line AB, we have only to recollect that B′C′ is the length of the projector of the point B; and that $b_{,}$A is the projection of AB. If, then, perpendicular to $b_{,}$A we make $b_{,}$K = B′C′, and join KA, the angle KA$b_{,}$ is the inclination, turned on the plane of projection.



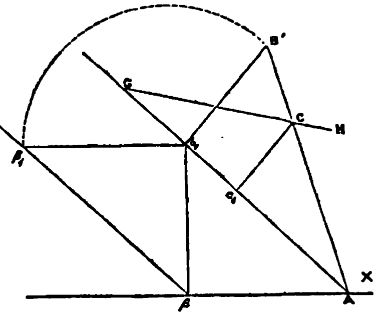
When one of the planes is at right angles to the plane of projection, the construction becomes simplified. For then the projection of the line of intersection falls upon the trace of that plane, as $b_,$A.

Also, for the inclination, take any point C in the other trace AC, and draw the perpendicular C$b_,$, and the parallel $b_,$H to AC; and make $b_,$CH equal to the inclination of that plane whose trace is AC. It cuts off $b_,$H the projector of the point B′ in the line of intersection. Whence we have only to take $b_,$B′ perpendicular to $b_,$A and equal to $b_,$H, and join B′A, giving the angle B′A$b_,$ as the inclination of the line of intersection.

PROPOSITION VII.

To find the projection and projector of the intersection of a given line with a given plane.

Let AX be the trace, and $b_,\beta\beta_,$ the inclination of the given plane, and let AG be the projection, G the trace, and HGA the inclination of the given line, it being turned with its projecting plane upon the plane of projection about its own projection AG.

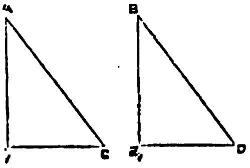


Then it is clear that the given plane and projecting plane are in precisely the same relation as in the second case of the preceding problem; and that AB′ is the line of their intersection after revolution. Whence C is the point of intersection of that line with the given line GH. Wherefore, $c_,$ is the projection, and C$c_,$ the projector of the point of intersection of the given line and given plane.

PROPOSITION VIII.

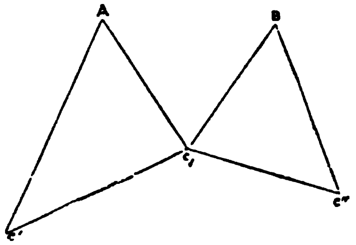
Two lines being given by means of their projections, traces, and inclinations to ascertain whether they be in the same plane or not.

(*a.*) Let the projections be parallel, as A$c_,$, B$d_,$: then if the inclinations be equal, the lines themselves are parallel, and in one plane. If the angles of inclination be not equal, they are not in one plane.



(*b.*) Let the projections intersect, as at $c_,$, and let A,B be their traces, and $c_,$AC′, $c_,$BC″ their inclinations: then drawing $c_,$C′ and $c_,$C″ perpendicular to A$c_,$, B$c_,$ respectively, if

$C'c_{,} = C''c_{,}$ they are in one plane; and if not, they are not in the same plane.



PROPOSITION IX.

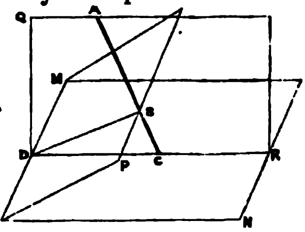
If a straight line be perpendicular to a plane, then:—

(1.) *The trace of the plane and orthographic projection of the line are at right angles to each other;*

(2.) *The projecting plane cuts the given plane in a line at right angles to the trace;*

(3.) *The intersections of the given plane and plane of projection by the projecting plane contain the angle of inclination of the given plane; and*

(4.) *The inclinations of the given plane and line are complementary.*



Let MN be the plane of projection, MP any other plane whose trace is MD, and ABC a line perpendicular to MP: then the projection CD of AC upon MN will be perpendicular to the trace MD.

For, let QR be the projecting plane of AC: then since it passes through AC perpendicular to MP, it is perpendicular to MP (*Pls. & Sols.*, *Chap.* II. 16); and it is by hypothesis perpendicular to MN. Whence it is perpendicular to their common section MD (*Pls. & Sols.*, *Chap.* II. 18). Hence, the angles MDC, MDB are right angles, which establish the first and second parts of the proposition.

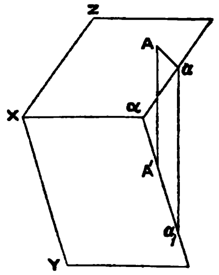
Again, since from D in the common section of MN, MP, the lines DC, DB are drawn perpendicular, one in each plane, the angle CDB is the inclination of MP to MN (*Pls. & Sols.*, *Def.* 4). This is the third property specified above.

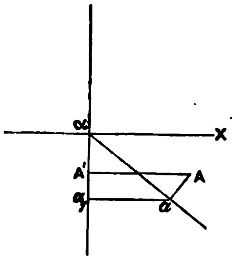
Lastly, since CD is the projection of AC, the angle ACD is the inclination of AC; and, as AC is perpendicular to the plane MP, the angle DBC is a right angle, and the remaining angles BDC, BCD (which are the inclinations of the plane and line) are equal to one right angle, or are complementary of each other.

PROPOSITION X.

A point is given in reference to a plane by means of its projection and projector, to find its projection and projector in reference to another given plane.

Let aZ be the plane to which the given point A is referred, and given by means of its projection a and projector Aa; and let YXZ be





the inclination, and αX the trace of the new plane upon which A is to be projected.

Since Aa is perpendicular to αZ, its projection on αY is perpendicular to the trace αX (*Pls.* II. 18), and hence the projections of the points A, a, in it are in a line perpendicular to αX.

Whence, since a is given in the given plane αZ, its projection a_1 on αXY can be found by a former construction (*Prop.* I.), and the line $a_1\alpha$ in which the projection of A upon αXY is situated is known.

Also since Aa is given in magnitude and position, and that aa_1 is found, it will follow that AA' parallel to aa_1 can be constructed; thus giving A' as the required projection of A upon αXY, AA' as its required projector, and $A'a_1$ as the projection of the line Aa.

Moreover, the system may be turned about aa_1 to coalesce with αXY the new plane of projection, and all the parts of the figure will retain their relative positions as regards aa_1 and each other. Wherefore, the entire data and constructive lines might have been *originally* exhibited there, and the point A' and the magnitude of AA' the new projector have been determined without any use of the figure in space at all. In fact, the figure actually used in the construction of the problem in practice (which is exhibited in the margin) is precisely that which the turning round of the figure in space produces.

SCHOLIUM. This problem is one of great importance in projection, being the foundation of the method of projecting figures of three dimensions. The figures are supposed to be given standing with one of their faces resting on the given plane αZ, whilst their several points which are not in that plane are supposed to be given by means of their projections on it and their corresponding projectors. These projectors are sometimes actual lines in he tproposed figure, as in the case of a cube or any right prism, but more frequently they are lines having definite relations to the figure, and which it will be necessary to determine *à priori* when the figure is defined independently of them. As exercises or examples of this kind, the platonic bodies, or regular solids, have generally been employed.

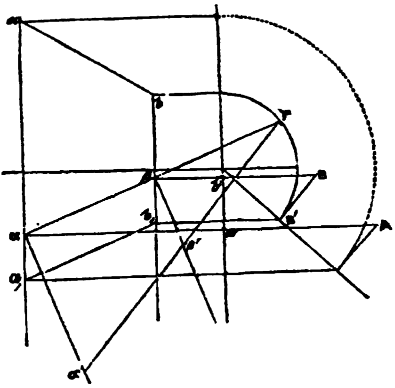
It is too obvious to call for special remark that a line is projected when two points in it are projected, and that its inclination becomes known when the projectors of three points are found. The same applies to all rectilinear and curvilinear boundaries, and lines drawn upon a given plane or surface. The entire projection of all bodies is thus comprised in this problem. It would lead us beyond moderate bounds to give any considerable number of examples; but it still appears incumbent on us to offer a few instructive ones in this place.

EXAMPLE 1.

Given two points in a line in reference to a given plane, to find the projection and inclination of the line in reference to the plane of projection.

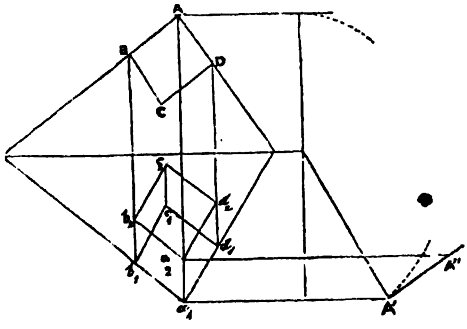
Let a, b be the projections of the points A, B on the given plane: project these in $a_{\prime}$, $b_{\prime}$ on the plane of projection; and project also A, B in α, β, and $\alpha\,\beta$ is the projection sought, and Aa', Bb' are the projectors on that plane.

Make $\alpha\alpha'$, $\beta\beta'$ perpendicular to $\alpha\beta$ and equal respectively to Aa', Bb': then $\alpha'\beta'$, $\alpha\beta$ being produced to meet in γ, we have $\alpha\gamma\alpha'$ the 'inclination of the given line to the plane of projection.



EXAMPLE 2.

A cube is given having one of its faces ABCD in a given position on a given plane, to construct its projection.



Project the square as before in $a_1b_1c_1d_1$. Project also the altitude A'A'', which is equal to one of the sides, giving a_1a_2: then a_2b_2, a_2d_2 being drawn parallel to a_1b_1, a_1d_1 will give b_2 and d_2 in the perpendiculars Bb_1, Dd_1; and lastly d_2c_2, b_2c_2 parallel to d_1c_1, b_1c_1 give c_2. The figure is then projected, as was required.

The same process applies to any right prism, whose base is situated on the given plane: but in case of such prism being cut obliquely, each of the edges must be separately projected, instead of the upper face being projected by parallels, as has been done in this construction.

EXAMPLE 3.

To construct a triangular pyramid, having its base given in a given plane, and likewise given the perpendicular from the vertex to the base in magnitude and position.

Find the projections of the base and of the vertex, and draw lines from the projections of the base to that of the vertex; and the projection is complete.

EXAMPLE 4.

Given an oblique prism in a given position on a given plane, to construct its projection.

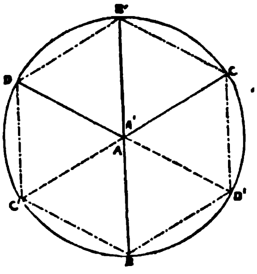
Project the base and one of the upper corners of the prism as in the preceding example; and then, as in the second, construct the projection of the upper face by parallels, and of the edges likewise.

EXAMPLE 5.

To project a cube on a plane at right angles to one of its diameters.

(*a*.) All the three edges which meet in one extremity of that diameter are equally inclined to the diameter, and hence, also, to the plane of projection: they will hence be projected into equal straight lines. Also, the three planes which connect these edges two and two, having their inclinations complementary to the inclinations of the opposite edges, will be equally inclined to the plane of projection; and hence their contained (right) angles will be projected into equal angles. These three projected angles, making together the angular space about the projection of the solid angle, are equal to four right angles; and hence each of them is 120°.

Let these be AB, AC, AD, of the annexed figure.

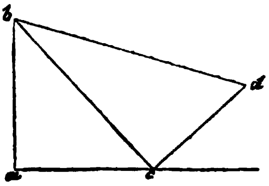


In the next place, the opposite solid angle of the cube is projected also in A (which we shall also mark A′), and the three edges which meet in this point are parallel to those already projected, but lying in the opposite side of the diagonal. These have the same inclination to the plane of projection that the former ones had: and hence their projections will be of the same length as the former ones, and be situated in continuation of them. They will therefore be represented by A′B′, A′C′, A′D′ respectively.

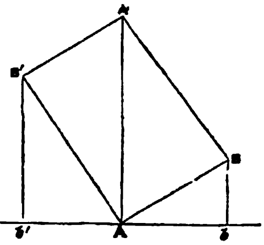
Lastly, the remaining edges being those which join the points already projected, we have only to join the adjacent projected points in succession, to obtain the projection of the entire cube.

It is obvious, therefore, that the projections of all the edges lie in the sides and diagonals of a regular hexagon; the edges which meet in the given diagonal forming the three diagonals, and the remaining edges the sides, of the hexagon.

This construction, however, merely gives the *form*, but not the *magnitude* of the projected figure, for any given length of edge. This relation is fixed, and is exhibited by a very simple construction.



Take two lines, *ab*, *ac* (Fig. 1), at right angles, and make each equal to the edge of the cube, and join *bc*: this will be the diagonal of one of the square faces. Make *cd* perpendicular to *bc* and equal to *ab* or *ac*, and join *bd*; this will be the diagonal of the cube itself.



In Fig. 2, draw AA′ perpendicular to the line *b′b*, and equal to the diagonal of the cube, *bd*. If, now, we conceive a plane to pass through the diagonal AA′ and one of the edges AB, it will be the projecting plane of AB, and A*b* will be its projection, as to *position*. Moreover this plane will pass through the opposite edge (since they are parallel and AA′ joins them), and will cut the faces adjacent to these edges at the points A, A′ in diagonals of those faces. Whence AB is an edge, A′B the diagonal of a face, and AA′ the diagonal of the cube itself. The three sides AB, BA′, A′A of the triangle AA′B are therefore given, being equal to *ba*, *bc*, *bd* of Fig. 1.

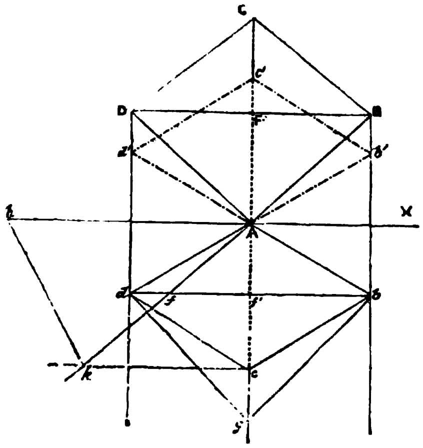
Wherefore, the entire construction will be:—

Construct figures 1 and 2 as above; then, with any centre and radius A*b* describe a circle; and in this circle describe a regular hexagon, as in the figure of the first analysis. This will be the projection sought.

(*b.*) The preceding construction is rather effected by special considerations belonging to the particular case, than by means of the general principle already developed, and to which the problem was proposed as an illustration. It may, therefore, be proper to give another which shall be more illustrative of the general principle.

In the first place, since each edge is perpendicular to the plane containing the other adjacent ones, that edge is perpendicular to the face which contains them: and hence the angle made by that plane with the plane of projection is the complement of that made by the edge with the plane of projection.

Let ABCD be a face of the cube; *h*AX the trace of the plane of



that face on the plane of projection. Make Ah the diagonal of the cube, Ak the diagonal of a face, and kh the edge of the cube into a triangle, and draw Ag perpendicular to the trace kX. Also, make Af equal to AF the semi-diagonal of the square face.

Then, as was shown in the concluding part of the former construction, kAg is the inclination of the face of the cube ABCD to the plane of projection; and by the general principle of the method, if we draw kc, and dfb, the figure $Abcd$ will be the projection of the face ABCD.

Again, hk is perpendicular to Ak, and equal to the edge CC′ of the cube which stands at right angles to ABCD at C; and hence the point C′ is projected in A. Also, the edges of the opposite face are parallel, each to each, to those of ABCD; that is, AB to A′B′, etc. Whence their projections will be parallel and equal to those of ABCD. We have, therefore, the parallalogram $Ab'c'd'$ as the projection of the face A′B′C′D′, equal and similar in all respects to $Abcd$.

Lastly, the projection of the edges joining these points two and two in order, will be the lines joining the projections of their extremities in the same order. Wherefore, drawing dd', bb', Ac, Ac', we have the entire projection sought.

All that was actually required by the problem to be done has been effected: but it may be desirable still to investigate the character of the figure into which the cube has been projected—that investigation to be founded on the construction itself as here effected.

By the construction the figure becomes a hexagon, with lines drawn to its angular points from a point within it; we shall show that it is a *regular hexagon*, and that each of the lines from the point within it is the radius of its circumscribing circle.

Since the six edges of the cube that meet, three in A and three in C′ are equally inclined to the diagonal AC′, they are equally inclined to the plane of projection, which is, by hypothesis, perpendicular to AC′. Whence they are projected into equal lines; or Ab, Ab', Ac, Ac', Ad, Ad' are all equal, and are situated, therefore, in the circumference of the circle about A.

Again, since AD, C′B′ are parallel and in the same projecting plane through the diagonal AC, they are projected in Ad, $b'A$ which form one continuous line. In like manner bA, Ad' are one line, and $c'A$, Ac are one line. The diagonals of the hexagon are, therefore, diameters of the circle about A.

Finally, since dd', Ac', $b'b$ and Ac are the projections of four parallel and equal edges of the cube, they are equal to each other; and hence the sides dd', bb' are each equal to the radius. In the same way each other pair of opposite sides may be proved respectively equal to the radius. Whence the figure is a regular hexagon.

If, as is sometimes the case, such a projection of the cube be given, and it is required to find the edge of the original cube, it may be easily effected as follows:—

Take any one of the equilateral parallelograms of which the hexagon is composed, as $Abcd$; draw the diagonals Ac, bd meeting in f, and make $fg = fd$: then dg is the edge of the original cube.

For, bd is the projection of the diagonal BD of the square face ABCD; and as this is parallel to the plane of projection, bd is equal to BD. Our construction, then, gives dg equal to the side of the square described on the diagonal *bd*; and which is therefore equal to the side of the square on the diagonal BD; that is, to the edge of the cube.

The same figure suggests also a ready method of finding the side of the hexagon when the edge of the cube is given, viz.:—

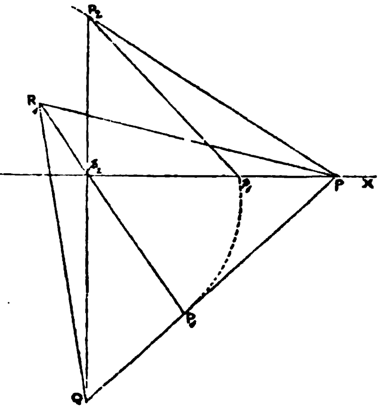
Let *dg* be the given side; on which describe an isosceles right-angled triangle *df′g*; and make the angle *f′dc* equal to 30°: then *dc* is the required length of the projection.

For, since A*bcd* is an equilateral parallelogram, *db* bisects the angle *d*, and hence, *f′dc* = 30°; and the construction is obviously true.

This projection is the foundation of the *Isometric* method of which further discussion will be given hereafter in the APPLICATIONS.

PROPOSITION XI.

Given the trace and inclination of a plane to a horizontal plane, to find its trace upon, and inclination to, a given vertical plane, and to construct the angle made by the traces of the plane upon the horizontal and vertical planes.



Let X′X be the given trace of the vertical plane upon the horizontal, and PP_1 the trace of the given inclined plane upon the horizontal.

(1.) Take any point p_2 in X′X, draw p_2 P_1 perpendicular to PP_1, and set off p_2p_1 on X′X (on either side of p_2) equal to p_2 P_1, and make p_2 p_1P_2 equal to the given inclination of the inclined plane to the horizontal, meeting the perpendicular from p_2 in P_2: then P_2 is a point in the trace of the inclined plane on the vertical plane through X′X. Also, P is another point; and hence, PP_2 is the required trace on the vertical plane.

For, since in the vertical plane, P_2p_2 is perpendicular to the trace in the horizontal plane, the line P_2p_2 is itself perpendicular to the horizontal plane; and hence the plane $P_2p_2P_1$ through it is perpendicular to the horizontal plane. Also since p_2P_1 is perpendicular to the trace of the oblique plane, and P_2p_2 perpendicular to the horizontal plane, the line P_2P_1 is perpendicular to PP_1 (*Pls. and Sols.* II. 13), and $P_2P_1p_2$ is the inclination of the oblique plane to the horizontal one.

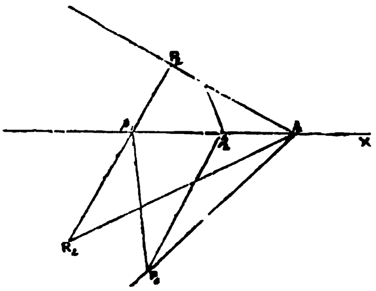
But in the triangles $P_2p_2P_1$, $P_2p_2p_1$, the angles $P_2p_2P_1$ and $P_2p_2p_1$ are right angles, and the side P_2p_2 common and p_2P_1, p_2p_1 equal; and hence the angles $P_2P_1p_2$, P_2 p_1p_2 are equal. Also $P_2p_1p_2$ was made equal to the given inclination; and hence the plane whose traces are P_1P and PP_2 has the given inclination to the horizontal plane. That is, PP_2 is the vertical trace required.

(2.) To find its inclination to the vertical plane. Take any point p_1 in X′X (Fig. 2), and draw p_1P_1 perpendicular to X′X to meet the horizontal trace in P_1, and p_1P_2 perpendicular to the vertical trace; make p_1 p_2 equal to p_1P_2 and join p_2P_1; then the angle $p_1p_2P_1$ is the inclination of the plane to the vertical plane.

For, as in the preceding reasoning, it may be shown that $p^1P_2\,P_1$ is the inclination, and that $p_1p_2P_1$ is equal to it.

(3.) To find the angle made by the traces themselves.

Conceive in Fig. 1., that the oblique plane is turned about PP_1 to coincide with the horizontal plane: then as P_2P_1 is perpendicular to PP_1 it will during this revolution always be perpendicular to PP_1; and hence when it coincides with the horizontal, will coincide with the perpendicular p_2P_1. Make, therefore, P_1R_1 equal to P_2P_1, and join R_1P. Then, evidently R_1PP_1 is the angle made by the traces of the plane.



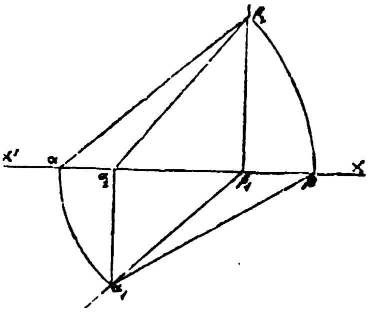
PROPOSITION XII.

Given the position of a line with respect to the horizontal plane, to find its position with respect to any vertical plane.

Let X'X be the trace of the vertical plane upon the horizontal one, $\alpha_1\beta_1$ the projection of the given line of which α_1 is the trace of the line on the horizontal plane, and β_1 the intersection of the projecting plane with the trace X_1X.

(1.) Then the projecting plane being also perpendicular to the horizontal one, its intersection $\beta_1\beta_2$ with the vertical will be perpendicular to X'X; and $\beta_1\beta_2$ will contain the trace of the given line upon the vertical plane.

Make $\beta_1\alpha$ equal to $\beta_1\alpha_1$, and the angle $\beta_1\alpha\beta_2$ equal to the given inclination: then β_2, the intersection, is the trace of the given line upon the vertical plane.



For evidently we have only turned the projecting plane about $\beta_1\beta_2$ to coincide with the vertical plane; and then turned the compound system about X'X to coincide with the horizontal plane.

(2.) Since the projecting plane of the given line upon the vertical is perpendicular to the vertical, its trace upon the horizontal will be perpendicular to X'X.

Whence, draw $\alpha_1\alpha_2$ perpendicular to X'X, which gives the projection of the point α_1; and we have β_2 already found, which being the trace of the line on the vertical plane, the line $\alpha_2\beta_2$ is the projection of the given line on the vertical plane.

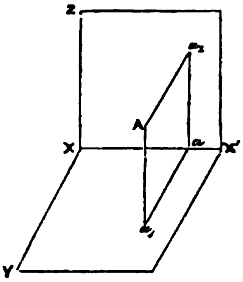
(3.) To find its inclination to the vertical plane. Make $\alpha_2\beta$ equal to $\alpha_2\beta_2$ and join $\beta\alpha_1$: then $\alpha_2\beta\alpha_1$ is the inclination of the given line to the vertical plane.

For $\alpha_1\beta_2\alpha_2$ is evidently the inclination in question, and $\alpha_2\beta\alpha_1$ is equal to it; the triangles $\alpha_1\beta_2\alpha_2$ and $\alpha_1\beta\alpha_2$ being in all respects equal.

PROPOSITION XIII.

A point is given in reference to the horizontal plane by means of its projection and projector; it is required to find its projection and projector in reference to a given vertical plane.

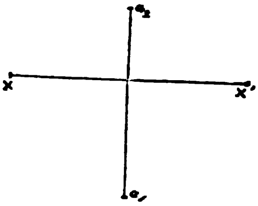
Let X'Y be the horizontal plane, a_1 the projection of the given point A, and Aa_1 its projector on that plane; also let X'Z be the given vertical plane, of which XX' is the trace.



Now conceive the projector on the vertical plane to be Aa_2 and a_2 the projection of the point A.

Then since Aa_1 and Aa_2 are perpendicular to the planes of projection respectively, the plane through them is perpendicular to each and to their common section XX' (*Pls.* II. 16, 18); and a_2a, a_1a are perpendicular to the planes X'Y and X'Z. Whence Aa_1 and a_2a are parallel, as are likewise Aa_2 and a_1a. Also, since Aa_1 is perpendicular to X'Y it is perpendicular to a_1a in it: and the figure $Aa_1\,aa_2$ is a rectangle, having a_1a equal to Aa_2 and aa_2 equal to Aa_1.

The projection, then, of A upon the vertical plane is in a plane through A perpendicular to the trace of the two planes of projection; its projector is equal to the given distance of a_1 from the trace, and its height above the trace is equal to the given projector upon the horizontal plane.



If now we turn the vertical plane about XX' to coincidence with the horizontal, aa_1 and aa_2 will form one straight line at right angles to XX'; and if we retain only these planes and the lines internally in them, the figure will be as in the margin.

We see then, that a_1 being given, we have only to draw a_1a perpendicular to XX', and make aa_2 equal to the given projector.

SCHOLIUM. We have already seen that when a figure is given for projection on any plane, as the horizontal, for instance, the plane to which that figure is referred being also given, we do actually find the lengths of the projectors themselves. (*See Props.* I.—IV.) We see by this proposition that the projections on a given vertical plane, when turned on the horizontal, are in lines from the horizontal projections perpendicular to the trace of the vertical plane; and that their altitudes above that line on the drawing are equal to the projectors of those points. Whence it becomes easy to construct simultaneously with the horizontal projection of a figure its projection on any given vertical plane.

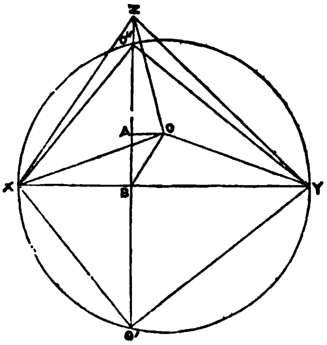
PROPOSITION XIV.

Let the three faces of a trihedral right angle be projected on any plane which also cuts those faces, and let those three triangular faces be turned about their traces on the plane of projection : then,

(1.) *The projections of the three edges will be respectively perpendicular to the traces of the opposite faces ;*

(2.) *They will respectively pass through the vertices (after revolution) of the opposite faces ;*

(3.) *The projections of the three edges will meet circles described on the three opposite traces in points which are the vertices of the three right-angled triangles, after revolution.*



Let O be the position of the vertex of the trihedral right angle, and XY, YZ, ZX the traces of its faces on the plane of projection XYZ. Let OA be perpendicular to XYZ, then A is the projection of the trihedral angular point O.

(1.) Then, since O is a trihedral right angle, OZ is perpendicular to each of the lines OX, OY, and therefore to the plane XOY which contains them. Also the projecting plane of ZO passes through AO, and is hence perpendicular to XYZ ; and it passes through OZ, and is hence perpendicular to XOY. (*Pls.* II. 16.) It is, therefore, perpendicular to ZXY and OXY ; and hence to their common section XY, and its traces ZB, BO are perpendicular to XY. The projection of OZ upon XYZ is therefore perpendicular to the trace of the face XOY upon XYZ, which is the first part of the proposition.*

(2.) In the next place, since BO is perpendicular to XY, it will remain so in its revolution on either side, as XO'Y or XO''Y ; and O will hence fall either in BZ or its prolongation, as O'' or O'. Whence the projections of the edges of the trihedral angle will pass through the vertices of the opposite faces when they are turned on the plane of projection.

(3.) The angle XOY being a right angle, its vertex will fall in the circle on XY ; and so of the others. This establishes the third part of the proposition.

SCHOLIUM. In contemplating this figure in space, and after revolution, several properties of it, having important practical uses, will present themselves to our notice.

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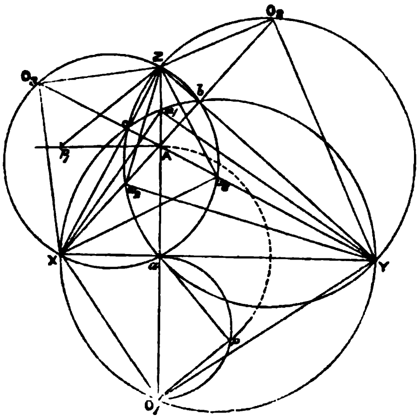
* This conclusion might be more simply deduced from *Prop.* IX., that since OZ is perpendicular to the plane XOY, the projection of OZ is perpendicular to the trace of XOY. In fact, the steps through which we have here passed are but a repetition of those by which that proposition was established ; and they are employed here chiefly with the view of more distinctly impressing that important though simple theorem on the student's mind.

We shall remark that in this figure there are concerned, besides the projection of O (which comprises, in fact, two data), the following triads:—



1. The three traces.
2. The three angles formed by them.
3. The lengths of the three edges.
4. The inclinations of the three edges.
5. The projections of the three edges.
6. The inclinations of the three facial triangles.
7. The three pairs of acute angles of the facial triangles.

Now, it requires but little acquaintance with the principles of plane geometry to perceive that if any five of these are given (except there be more than three in the first pair of triads) all the rest can be found by very simple constructions. In the annexed figure, the revolutions are performed for the three facial right-angled triangles, O_1, O_2, O_3, being the positions of the vertex O, when all fall without the triangle, and ω_1, ω_2, ω_3, those when all fall within it; A is the projection of O, and XA, YA, ZA those of the edges respectively; and the circles have their second sections *a*, *b*, *c* in the traces XY, YZ, ZX, since the lines X*b*, Y*c*, Z*a* are perpendicular to the traces.

Let us take an example:—

Given two traces XY, YZ, *the angle* XYZ, *and the projections of two of the edges* OX, OY.

With the first three of the data form the triangle XYZ; and with base XY and the given projections, the triangle XAY. On the three sides of XYZ describe circles and draw the other lines as indicated in the figure.

Then XO_1Y, YO_2Z, ZO_3X, or $X\omega_1Y$, $Y\omega_2Z$, $Z\omega_3X$ will be the three facial triangles turned on the plane of XYZ; and ZA will be the projection of the third edge.

The projections of the lines in which the projecting planes cut the facial planes, will be A*a*, A*b*, A*c*.

The length of the projector may be thus found:—on aO_1 describe a semicircle apO_1, and in it set off $ap = Aa$; then, obviously, the triangle apO_1 of the figure is in all respects equal to the triangle aOA of the figure in space, and hence O_1p is equal to the projector. It is also clear that the construction for O_1p might have been made upon any of the corresponding lines of the figure, should greater practical convenience result from such a change.

Again, since the angle OaA is the inclination of XOY to XZY, it is found in the preceding construction to be O_1ap.

Lastly, the inclinations of the edges will be found by taking (for instance, that of OZ) Ap_1 perpendicular to AZ and equal to O_1p, and joining p_1Z. Then p_1ZA is the inclination of OZ to XYZ.

This figure was proposed by the late Mr. Nicholson, which he called the *directing diagram*. We shall distinguish it as *Nicholson's diagram;* and some applications of it to practice will be given in a future part of this work. The student is recommended in the present stage to vary the data so as to present new problems for exercise; as a familiarity with the diagram will prove of great use to him in several branches of his future practice.

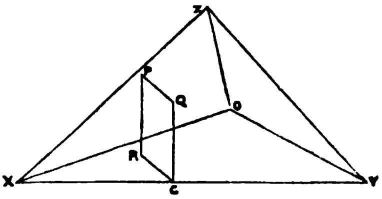
PROPOSITION XV.

If a point be projected upon three rectangular planes, and likewise upon any other plane which cuts them, and if the three planes be made to revolve as in the preceding proposition, and lines be drawn through the three projections, after revolution, perpendicular to the traces respectively;

(1.) *These lines will pass through the same point;*

(2.) *That point will be the projection of the original point on the plane which cuts the three rectangular planes.*

Let P be the point whose projection on XYZ is Q, and whose projection on XOY is R; draw the plane through PQR, which will be perpendicular to both planes XZY and XOY, being drawn through the projectors PQ, PR. Whence QC, RC are both perpendicular to the trace XY; and when XOY is turned on the plane XYZ, QC, CR will be in one line perpendicular to XY. The projection R, then, is such that a perpendicular being drawn from it to the trace, it will pass through Q the projection of the point P upon XYZ. And similarly for the other planes.



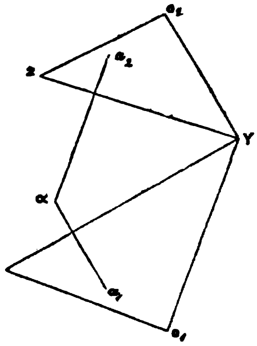
PROPOSITION XVI.

The projections of a point on two given rectangular planes are given, to find its projection on a third given plane.

This problem naturally divides itself into two general cases, and each of these into subordinate ones. The general cases are—

(*a*.) When the rectangular planes are given by their traces on the third plane.

(*b*.) When the third plane is given by its traces on the two rectangular ones.

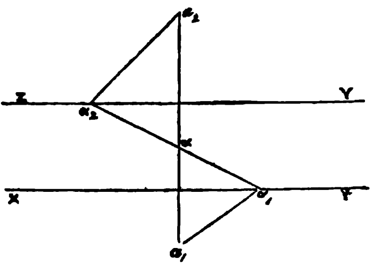


(a_1.) Let XY, YZ be the traces of the rectangular planes upon the third given plane, and a_1, a_2 the projections of the point on those planes, given with respect to any two lines in those planes respectively.

Then the rectangular planes being made to revolve about the traces till they coincide with the third plane, draw $a_1\alpha$, $a_2\alpha$ perpendicular to the traces, meeting in α; and α will be the projection required.

For by the preceding proposition, both these perpendiculars pass through the projection of the point on XYZ.

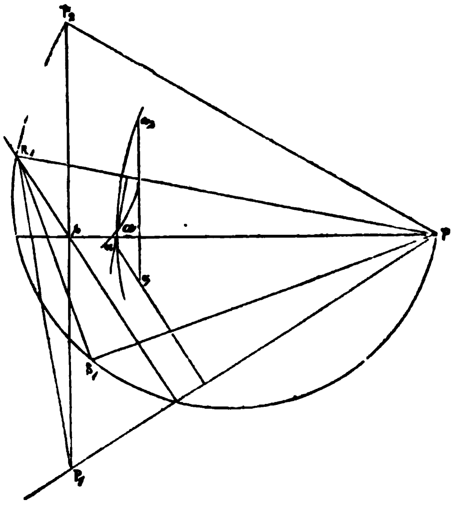
(a_2.) When the traces are parallel, these lines coincide in direction, and the preceding construction fails in this case. We may, however, proceed thus:—



Draw $a_1\alpha$, $a_2\alpha_2$ parallel to each other meeting the traces in α_1, α_2: then a_1a_2, $\alpha_1\alpha_2$ will intersect in α the projection sought.

(b_1.) Given the projections of a point a_1, a_2, on the two rectangular planes P_1Pp, P_2Pp; and the traces of the plane P upon which the projection is to be made; to find the projection, when the plane P is turned about its trace on one of the given rectangular planes (Fig, 3).

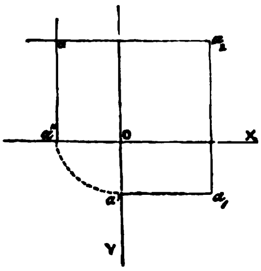
Cut the system of planes by a profile to the given rectangular ones, and construct the triangle P_1PP_2 (*Prop.* XI.), on the horizontal plane, suppose, and let it be P_1PR_1.



We may now consider that one of the rectangular planes is turned upon P_1PR_1; and we have only to construct upon R_1P the right angled triangle R_1S_1P equal to P_2pP, and place a in it as a_2 is placed in P_2pP. Then we have, evidently, the problem reduced to the precise state of (a_1); and for similar reasons, it is only required to draw perpendiculars from a_1 to PP_1 and from a to PR_1, intersecting in α the projection sought.

In the actual construction of the problem, it is only necessary to construct PR_1 and a; the other parts of the figure being merely illustrative.

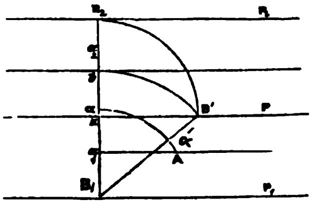
$(b_2.)$ When the third plane is a profile plane, the steps are more simple.



Draw $a_2\alpha$, a_1a' parallel to the axis; make $Oa'' = Oa'$, and draw $a''\ \alpha$ parallel to YZ, meeting $a_2\alpha$ in α: then α is the projection on the profile plane, when turned on the vertical plane.

If the horizontal projection be required, reverse the operations with respect to a_1 and a_2.

$(b_3.)$ Let B′P be the axis, B_2P_2, B_1P_1 the traces of the third plane parallel to the axis aP; and let a_1, a_2 be the projections of a point on the rectangular planes: it is required to find its projection on the third plane.



The projection will be on the line in which the profile plane through the given point cuts the plane of projection. Produce the line a_1a_2 to meet the traces in B_1, B_2 and the axis in a; make aB′ = $a\mathrm{B}_2$ and join B_1B′, then B_1B′ is the intersection of the profile and third plane of projection, turned on the horizontal plane. Make a_1A equal to $a\mathrm{B}_2$ and parallel to aP: then A is the given point turned also on the horizontal plane. Draw Aα' perpendicular to B_1B′; then α' is the projection on the horizontal plane. Make $B_1\alpha = B_1\alpha'$: then α is the point required on the plane P_1P_2 turned about B_1P_1.

The trace of the third plane on the vertical plane, after revolution, is found by making $B_1b = B_1B'$, and drawing the parallel to aP through b.

PROPOSITION XVII.

If a point be projected upon any three planes which meet, and these planes with the projections respectively upon them be turned on a fourth plane which cuts those three, about their traces, then perpendiculars from the three projections to the traces will all pass through one point, the projection of the given point upon the fourth plane.

For it follows from *Prop.* xv. that each of the three projectors of the point upon the three given planes does so; and hence it follows that all do so.

SCHOLIUM This is, obviously, a more general theorem than that which forms the basis of *Nicholson's theorem* in the preceding propositions. All projections, indeed, by means of plan and section on rectangular planes, are capable of being made with facility by that method: still this more general form will sometimes simplify the actual process to a considerable degree, since we may take these three planes to represent any trihedral angle of the figure to be projected.

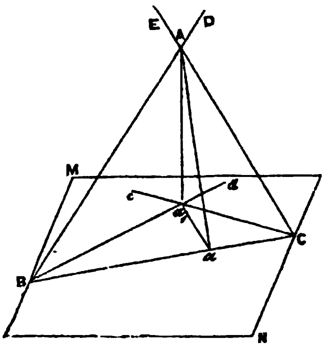
PROPOSITION XVIII.

To project a given angle.

Project the containing sides: then the projections will, obviously, contain an angle which is the projection of the given angle. The construction of this problem has, in fact, been already ***virtually*** given: but as there are some facts in connexion between the given angle and its projection which it is important to state specifically, the problem is enunciated here for the purpose of noticing them at length.

1. If both lines AB, AC be parallel to the plane of projection, their projections $a_{,}b_{,}$, $a_{,}c_{,}$ will be respectively parallel to themselves; and hence the lines and their projections will contain equal angles (*Pls. & Sols.* I. 7.).

2. Let both lines cut the plane of projection, as in B and C; and draw A$a_{,}$ from the angular point A perpendicular to the plane of projection MN; and join B$a_{,}$, $a_{,}$C. Then B$a_{,}$C is the projection of BAC. But (*Pls. & Sols.* III. 5) the angle B$a_{,}$C is greater than BAC. Wherefore, the projection of the angle formed by AB, AC and opposite to the trace of their plane on the plane of projection, is greater than the angle itself.



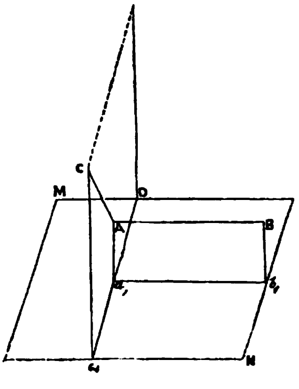
The vertical angle EAD to BAC is projected into the angle $ea_{,}d$ vertical to B$a_{,}$C, and hence also the same holds true respecting these.

But if one branch of the line be estimated towards the trace, and the other in the opposite direction, as CA, AD, then the projected angle C$a_{,}d$ is less than the given angle CAD: these being the respective supplements of the angles to which the preceding demonstration was applied, and the greater angle having the less supplement.

3. Let one line be parallel to the plane of projection, and the two form a right angle; then the right angle is projected into a right angle.

Let AB be parallel to the plane of projection and the angle CAB a right angle: then its projection will also be a right angle.

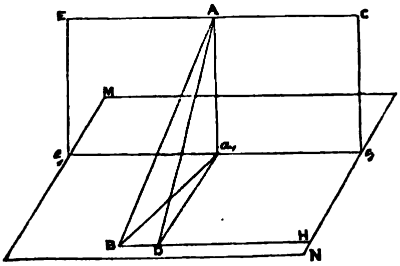
Draw the plane CD through A perpendicular to AB: then it will be perpendicular to the plane MN (*Pls. & Sols.* II. 19). It also passes through AC since all the perpendiculars through A to AB are in a plane perpendicular to AB. Whence CD is the projecting plane of AC, and $a_{,}c_{,}$ is its projection on MN.



Now the projection $a_{,}b_{,}$ of AB is parallel to AB itself, and AB is perpendicular to the plane CD: hence also $a_{,}b_{,}$ is perpendicular to CD, and therefore to the line $a_{,}c_{,}$ in it. Whence the projections $c_{,}a_{,}$, $a_{,}b_{,}$ of the sides of the right angle CAB form, themselves, a right angle $c_{,}a_{,}b_{,}$.

4. Conversely, if the projection be a right angle and one of the sides of the angles itself be parallel to the plane of projection, the angle itself which was projected is a right angle.

5. If one leg of the angle be parallel to the plane of projection, the angle is projected into an obtuse angle if itself be obtuse, and into an acute angle if itself be acute; and the projecting planes will form an obtuse or an acute angle in the two respective cases.



Let AC be parallel to the plane MN of projection, CAB be an obtuse angle, and B the trace of the leg AB upon the plane MN; and let B$Aa_{,}$ and A$c_{,}$ be the projecting planes of BA, AC; and B$a_{,}$, $a_{,}c_{,}$ the projections: then B$a_{,}c_{,}$ will be an obtuse angle, and the planes B$Aa_{,}$, C$Aa_{,}c_{,}$ form an obtuse angle.

For let the plane BAC be drawn: its trace on the projecting plane MN is a line parallel to AC and its projection $a_{,}c_{,}$ (*Pls. and Sols.* I. 2): also draw, in the plane BAC, the line AD at right angles to AC, meeting the trace of the plane BAC in D, and join D$a_{,}$.

Then, since A$a_{,}$ is the intersection of the projecting planes of BA, AC it is perpendicular to MN; and hence D$a_{,}$ is the projection of DA, and (*Case* 3) D$a_{,}c_{,}$ is a right angle. But B$a_{,}c_{,}$ is greater than D$a_{,}c_{,}$; and therefore the projection is an obtuse angle.

Also, since B$a_{,}$, $a_{,}c_{,}$ are perpendicular to the common section A$a_{,}$ of the projecting planes, the angle B$a_{,}c_{,}$ is the measure of the angle formed by those planes; hence the inclination of the projecting planes to one another is obtuse.

Again, produce CA, $c_{,}a_{,}$ to E and $e_{,}$ respectively; then since BAC, B$a_{,}c_{,}$ are obtuse angles, their supplements BAE, B$a_{,}e_{,}$ are acute; and B$a_{,}e_{,}$ is the projection of BAE. Also the angle B$a_{,}e_{,}$ is the measure of the inclination of the planes E$Aa_{,}e_{,}$, A$a_{,}$B to each other, and hence the inclination of the projecting planes is acute.

Both conclusions are, therefore, established.

EXERCISES ON ORTHOGRAPHIC PROJECTION.

1. Given a triangle and the inclinations of two of its sides to the plane of projection, to find the inclination of the third side and of the plane which contains the triangle to the plane of projection.

2. An angle is given, and its containing sides make given angles with the horizon; it is required to construct its projection on the plane of the horizon, or "to reduce the given angle to the horizon."

3. Given an angle BAC to construct a plane upon which the projection of that angle shall be a given angle, whilst the trace of that plane shall be a given line in the plane of the angle BAC.

4. Given the projections and projectors of three angular points of a parallelogram, to find those of the fourth.

5. A circle whose radius is γ is projected on a plane whose trace is given, and this projection again projected on a second plane, which is to be determined, when it again becomes a circle whose diameter is γ; find the inclination of the line in which the third plane intersects the first, to the second plane.

6. Given three points, D, E, F, in reference to the plane of projection, through which three sides of a triangle ABC are to pass, and given, likewise, the inclinations of the three sides of the triangle to the plane of projection; to construct the triangle, its projection, and the inclination of its plane to the plane of projection.

DESCRIPTIVE GEOMETRY.

In the investigations respecting figures in space, the main, or indeed the only, real object kept in view has been to deduce the geometrical properties of the lines and planes formed according to the hypotheses of the propositions. For this purpose the method of proceeding which we have employed is not only the most obvious and natural, but the most simple in conception and the most easy of development; and were the deduction of speculative truths the only purpose of our researches, there would be no necessity for the adoption of any other method, at least as far as this class of figures was concerned. However, as in plane geometry, so also in that of space, the ultimate object at which we aim, in reference to social life, is of a *practical* character; and it hence becomes necessary to accommodate our researches to meet the social wants created by the arts, whether of the painter, the architect, the engineer, or the artisan.

It requires but little consideration to convince us that beyond the deduction of speculative truths the method employed by Euclid with respect to the geometry of space, is unsuited to this purpose: for it implies the construction of lines and planes, and the determination of points which cannot be *practically* fulfilled. Neither, indeed, is it proposed to actually fulfil these conditions by the following methods; but to substitute for the proposed operations certain others, by which the same *constructive results* are obtained as would have been obtained by the theoretical constructions of the old geometry could they have been actually performed. Moreover, to render this object really attainable, it is quite clear that the entire series of operations to be actually performed must be performable upon a single plane, represented by the paper upon which the drawing is made. That this can be accomplished will presently be seen.

One mode of accomplishing this has, indeed, been already shown, under the head of Orthographic Projection. This is, in reality, the foundation of all methods, however different in appearance they may be in their ultimate development and practice. Besides the perspective projection, there are two methods of applying the orthographic projection effectively in practice: the *Descriptive Geometry* and the *Method of Contours*. A still further subordinate mode of application has of late years been brought into use in this country (and in this country only) under the title of *Isometric Projection*, which is found to be very serviceable in architectural and engineering drawings. Each of these subjects will be here taken in succession, so as to render each auxiliary to the study of the succeeding ones.

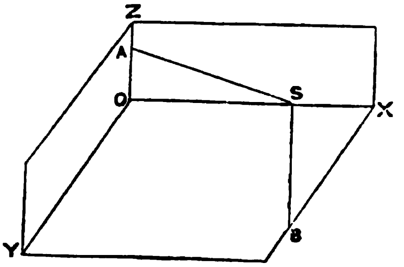
SECTION I.

DEFINITIONS.

Let us conceive any two planes which intersect each other to be given in a fixed position, and making any dihedral angle with each other; then all other figures in space are said to be *given* when the positions of their defining parts with respect to these two planes are given.

Def. 1. These planes are called *coordinate planes*, and their intersection the *axis*.

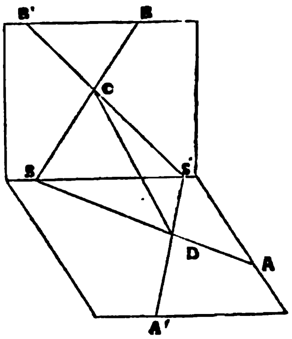
Def. 2. If a plane be given with respect to the coordinate planes, it will cut them in two lines (generally) AS, SB, which are called the *traces of the plane*, upon the planes ZOX, YOX.



Now when the traces are given, the plane itself becomes fixed; for no other plane than that one can pass at the same time through *both* the traces; and the *two* traces are essential to the definition of the plane since one alone might be common to innumerable planes, and could not therefore express one individual of the series.

Def. 3. Two planes intersect in a straight line; and hence if the two planes be given, the straight line will also be given. Let, then, the two planes be given by their traces BS, SA and B'S', S'A': then the intersections of the traces D and C will give the points in which the coordinate planes are cut by the line itself. These points are called the *traces of the line;* and they define its position completely, since no other line could pass through those two points.

Def. 4. A point not in either plane cannot be directly exhibited: but we may conceive it given as the common section of three planes, whose traces are given; or as the intersection of two lines whose traces are given. This latter form for the point is analogous to the definition of a line by means of two planes. We shall presently consider a modification of this.

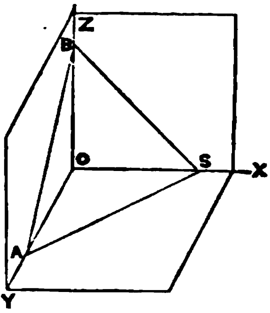


Def. 5. Instead of taking the *angle of ordination* (the dihedral angle of the coordinate planes) as any whatever, most of the constructions and the whole of the reasonings are much simplified by taking it

rectangular: and this plan is universally adopted in the processes of Descriptive Geometry.

The plane XY is taken horizontal, and XZ vertical; and are respectively called the *horizontal plane*, and the *vertical plane*; and the section OX is called the *axis*, or the *ground line*.

Def. 6. In some cases it is advantageous to take a plane perpendicular to the axis, *viz.*, YZ *instead of* one or other of the planes XY, XZ. This, however, will be noticed hereafter; and it is only necessary to remark, that no *additional* information is afforded of the position of the plane by the introduction of the plane ZY into the figure; since it is already given by means of the two traces AS, BS; and it is clear that two traces upon any pair of planes gives the trace upon the third.



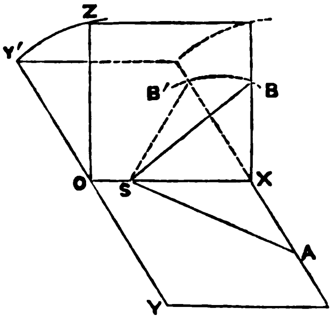
The three planes are very often called by technical names taken from the arts: XY, the *plan*; XZ, the *elevation*; and ZY the *section* (or profile): and as they are of perpetual occurrence in the application of Descriptive Geometry, we shall employ them from the outset; and when it is not otherwise expressed, we shall consider the two planes we use as the horizontal and vertical planes, that is, the plan and elevation.

The primary principle in Descriptive Geometry is, to perform *all* the operations of solution upon these two planes, whilst every operation upon each shall be *successive* to an operation upon one or other plane, and in no case require simultaneous use of both. At the same time, for the mere purpose of demonstration, we may make any conceivable hypothetical construction though not upon these planes, as in the geometry of Euclid's 11th Book. That the data, as far as planes, lines, and points are concerned, may be exhibited in this way we have already seen; and upon these planes also must the quæsita be finally exhibited as well as found. When other figures than these are concerned, the principle is the same; but this will be better understood when we have completed the present section of our course.

If these conditions can be fulfilled, it can evidently make no real difference in the two parts of the drawing (the plan and elevation) in what position they stand with respect to each other; and hence that one part may be removed to any other position that may be convenient in a practical point of view. One of the most obvious changes of place, and that which is found to be practically *most convenient*, is to cause one of the planes to revolve about the axis or ground line till it coincides with the extension of the plane of the plan beyond the ground line. This is the method universally adopted.

We might, however, have revolved the plan round the ground line till it coincided with the downward extension of the elevation; and no difference of general appearance or relation would exist in the figure in one method from that in the other. In one case we should have both upon the same horizontal plane, and in the other both upon the same vertical plane.

This is best illustrated by a model where the relative positions of the parts are better seen than in a mere pictorial representation.



In this motion every point in XZ describes the quadrant of a circle; and all the figures traced upon that plane retain their original magnitudes and relation of their several parts to one another. In the figure, SB′ represents SB the vertical trace; and it is easy to see that however it may change its *position* with respect to the other trace SA, it does not change with respect to the axis OX; and hence, so long as the relation of SB to SA is independent of the angle formed by them in space, no constructive change results from this supposed revolution.

Def. 7. The following terms are used in reference to this revolution of the coordinate planes :—

(1) To *plan the elevation*, is to bring the elevation into continuity with the plan;

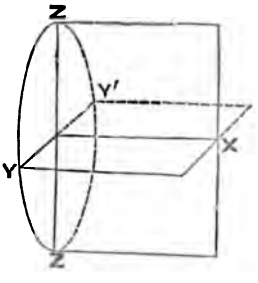
(2) To *elevate the plan*, is to bring the plan into continuity with the elevation;

(3) To *plan the section*, is to bring the section by revolution round OY, into continuity with the plan.

Def. 8. We shall designate the geometry of Euclid by the term *Theoretical Geometry*, to distinguish it from the *Descriptive Geometry* of which we here treat.

Def. 9. The model, or the representation of the figure as it exists, and as we contemplate it in Theoretical Geometry, we shall call the *eidograph*, or likeness; and the drawing which we execute as its substitute in Descriptive Geometry, from the one part being placed at right angles to its natural position, the *orthograph*.

Def. 10. By the *regions* of space, we mean those portions of space which are separated by the coordinate planes from each other, and of which these planes are the limitations; these planes being indefinitely continued in their respective directions.



The first region is that *above* the H and *before* the V (H and V denoting the horizontal and vertical planes); the second, *above* H and *behind* V; the third, *below* H and *behind* V; and the fourth, *below* H and *before* V. The regions being in succession from the spectator in revolving round the axis.

One or two other definitions will be given further on, when they will be better comprehended than at present.

In Descriptive, as in Theoretical Geometry, the properties of the eidographic figure, as well as of the orthographic, must be investigated; for the eidographic properties are always the authorities for orthographic practice. This, of course, will often require us to give the figures of the problem in both kinds; and it will, for the most part, be found that those properties are not only most directly but most easily obtained from the eidograph.

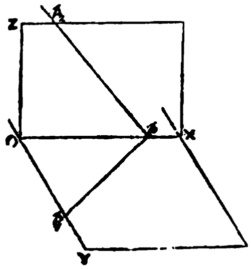
THE TRACES OF PLANES.

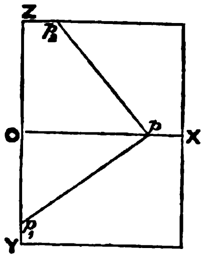
It is important to get a clear notion of the positions of the traces for particular positions of the planes; and conversely, of the planes themselves from the positions of the traces. For this purpose we give, attached to a verbal description, the figures both in eidograph and orthograph, of each peculiarity of case.

We shall minutely describe them in reference to the first region of space; and then only give the general representation of the cases for the other three regions.

VERBAL DESCRIPTION. EIDOGRAPH. ORTHOGRAPH.

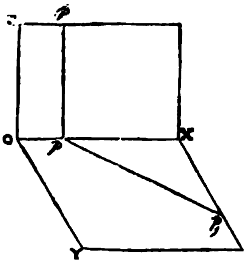
1. If a plane meet the axis, then both the traces meet it in the same point; and, if one trace meet the axis, the other meets it in the same point, and the plane itself meets the axis in that point.

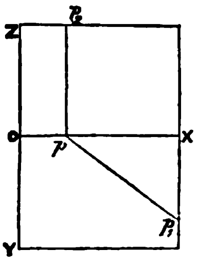


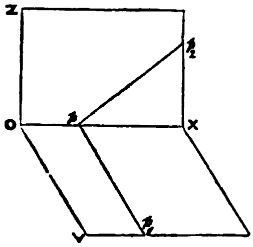


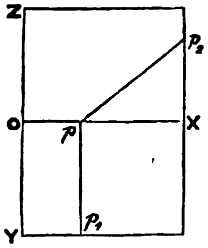
2. When the plane is perpendicular to one of the coordinate planes, its trace upon the other is perpendicular to the axis.

(In the first pair of figures the plane is perpendicular to the plan; and in the second pair, perpendicular to the elevation.)

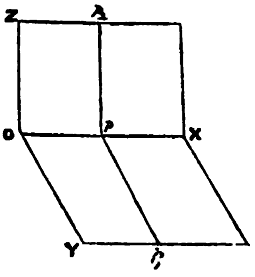


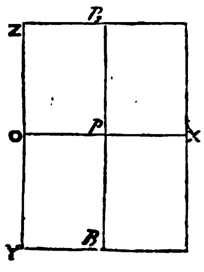






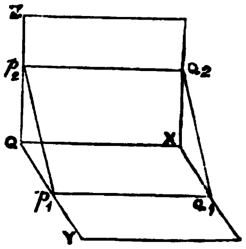
3. If the plane be perpendicular to both coordinate planes, their traces in eidograph are at right angles, and in orthograph in one straight line at right angles to the axis.

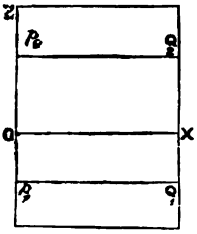




VERBAL DESCRIPTION. — EIDOGRAPH. — ORTHOGRAPH.

4. If the plane be parallel to the axis, but not to either coordinate plane, the traces will also be parallel to the axis.





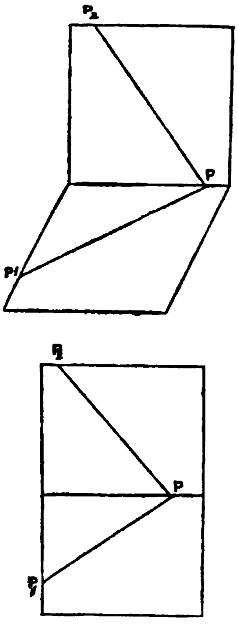
5. If the plane be parallel to one of the coordinate planes, there will be no trace upon that plane, and the trace upon the other will be parallel to the axis.

Though in these figures the planes are marked with boundary lines, in conformity with our practice in theoretical geometry, it is to be understood that in all cases they are capable of unlimited extension in all their directions. In the orthographs, it is not usual to make them at all, further than by the axis drawn horizontally on the paper, the boundaries of the paper being taken as the marked boundaries of the plane on which the orthograph is made. The traces, moreover, instead of being continued to the boundaries of the paper, are only drawn so far as the practical constructions in the individual case require them to be.

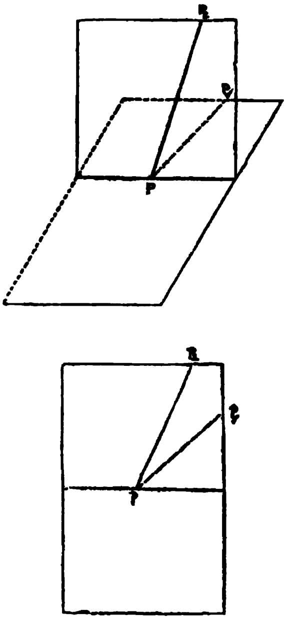
In all the figures which we have given, the traces of the plane separate the parts of it in *front of the elevation* and *above the plan*, from the parts of it which are concealed behind the one and below the other. We can generally choose the particular parts of the coordinate planes in reference to our drawing, as to fulfil this condition; but where it is for any reason desirable to have it otherwise, a system of notation expressive of *position* will enable us to express the parts of a figure behind the elevation, below the plan, or both.

It will be observed that we have marked the intersection of the delineated plane with the axis, both in eidograph and in orthograph, by the letter P; and any point in the horizontal trace by P_1, and in the vertical trace by P_2. This plane we shall briefly term "the plane P." In like manner, if more planes be drawn, we shall mark them by other letters Q, R, etc., plain, and with subscribed numerals taken in the same way, Q_1, R_1, signifying any point of their respective horizontal traces, and Q_2, R_2, etc., points in the vertical traces of the planes, Q, R, etc. When, therefore, we mark by literal notation, the signification will be apparent at once.

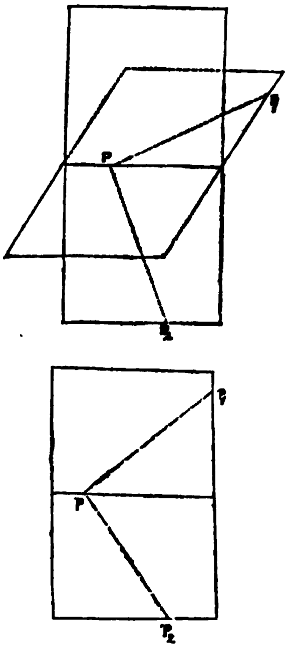
But as we often draw our orthographs without letters, the method of using round small dots (instead of full line, as in the preceding diagrams), to represent the parts of the traces which lie behind the vertical plane of the eidograph and below the horizontal plane, is found to fully answer the purpose in view. It is only necessary to keep in memory, that when the vertical plane is turned about the axis on the horizontal, the part of it below the axis comes to the front of the axis in the orthograph. Hence the following figures will be intelligible and distinguishable, whether considered in full line and lettered, or in dotted line without letter:—



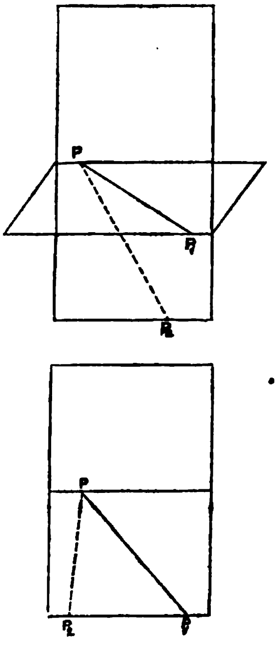
1st Region.



2nd Region.



3rd Region.



4th Region.

All these, obviously, are perfectly distinct by means of *either notation.*

All the varieties delineated in the first region may thus have correspondent varieties delineated in any region, either by dotting or by literal designations.

THE ORTHOGRAPH DELINEATION OF LINES.

It has been already noticed that the most convenient method of considering a line, in reference to Descriptive Geometry, is as the intersection of two planes; and that when the traces of the two planes passing through it are given, those of the line itself (or the points in which it cuts the coordinate planes) are also given. We have, hence, a choice of two methods of representation :—

(1.) *The traces of the two planes which pass through it;*
(2.) *The traces of the line itself.*

The circumstance that the line itself whilst absolutely fixed, may have innumerable planes drawn through it, shows that the same line may be represented by the traces of the planes in innumerable different ways; the only condition to be fulfilled being that the vertical traces of these planes shall all pass through one given point on the elevation, and the horizontal through one given point on the plan. This has the advantage of allowing us to make any hypothesis we please respecting the *positions* of the defining planes, compatible with that twofold condition; and we shall presently see that the one most generally advantageous is, to take the planes *perpendicular to the coordinate planes.* The trace of the plane perpendicular to the plan upon the plan is then called the *horizontal projection* of the line; and that of the plane perpendicular to the elevation upon the elevation is called the *vertical projection* of the line. The planes themselves are called the *projecting planes.*

However, in the mere exhibition of a line which is *given*, it is usually most convenient to define it by its traces;* as any other proposed mode of connecting it with the other data is at once obtained from these.

The following cases are enunciated in reference to their eidographs, from their being those which most usually occur in actual constructions for practical purposes. In passing from the eidograph, the line itself is of course omitted, its traces, or the traces of its defining planes, being all that our object requires us to use.

The following are the only possible positions of lines in reference to the coordinate planes:—

1. Oblique to both coordinate planes;
2. Oblique to one and parallel to the other;
3. Parallel to both planes; and, consequently to the axis;
4. Perpendicular to one plane, and therefore parallel to the other.

We shall take these four classes in succession, and give specimen eidographs as before; and in the figures it is to be understood that the parts marked in clear line are in the dihedral region in front of the elevation and above the plan; whilst the dotted parts are behind the elevation and below the plan in precisely the same way as has been shown with the plane. This will sufficiently indicate position to the eye alone, without further special or verbal description.

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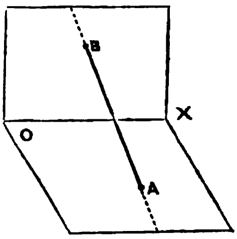
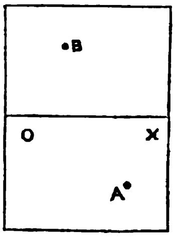
* When, however, the line is parallel to one of the coordinate planes, it makes no trace upon that plane; and when parallel to the axis, it makes no trace upon either plane. In this case we have no alternative but to define the line by planes through it; and the projecting planes are for the most part employed for this purpose.

CLASS 1. Lines oblique to both coordinate planes.

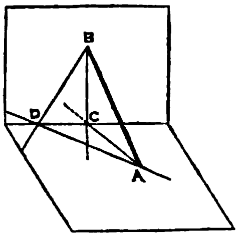
VERBAL DESCRIPTION. EIDOGRAPH. ORTHOGRAPH.

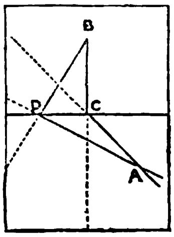
1. The line cuts the coordinate planes in the two points A, B; and is represented in orthograph by the points A, B, marked on the respective coordinate planes.

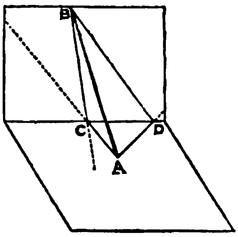


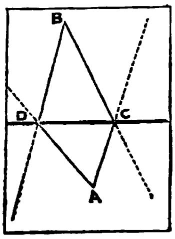


2. One plane perpendicular to a coordinate plane (in the first pair perpendicular to the plan, in the second to the elevation), and the other plane perpendicular to neither coordinate plane.



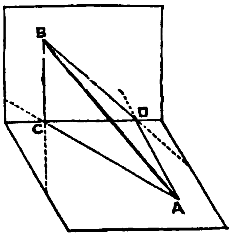


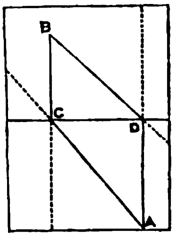




3. Both planes perpendicular to the coordinate planes, one to one, and the other to the other.

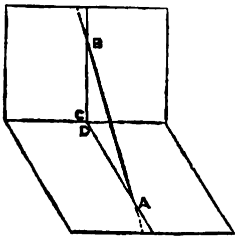
This is the system of planes most commonly used; and AC, BD are called the *projections* of the line BA upon the plan and elevation respectively.

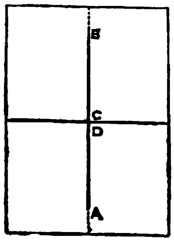




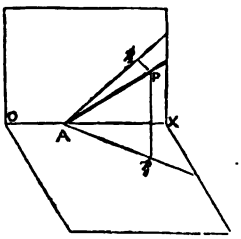
VERBAL DESCRIPTION. **EIDOGRAPH.** **ORTHOGRAPH.**

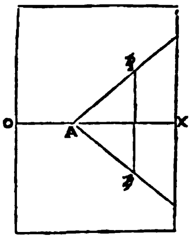
4. In the particular case in which the line AB is perpendicular to the axis (that is, when a line parallel to AB through a point in the axis would be perpendicular), the method of projection will express it only when we also mark the traces A,B; for the two projecting planes then coalesce and only give a single pair of traces. It will be as annexed.





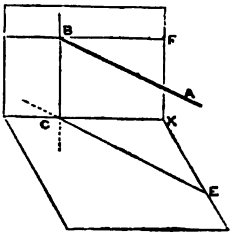
5. If the line pass through the axis and be oblique to both planes, the projecting planes will also pass through that point in the axis, and being perpendicular to the planes, the two points of the projections which correspond to any point in the line will be in a line perpendicular to the axis.

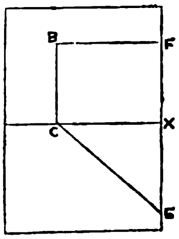




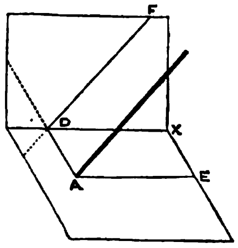
CLASS 2. Lines oblique to one, and parallel to the other coordinate plane.

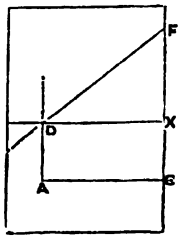
1. Parallel to the plan and oblique to the elevation. In this, CE is parallel to the given line AB, and BC is perpendicular to the axis, and the vertical projection BF is parallel to the axis.





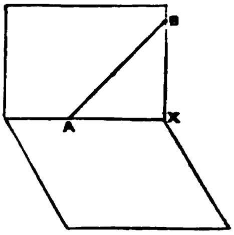
2. Parallel to the elevation and oblique to the plan. Corresponding in all respects to the other except the exchanged position.

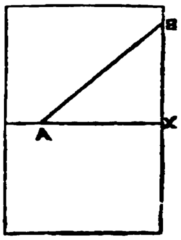




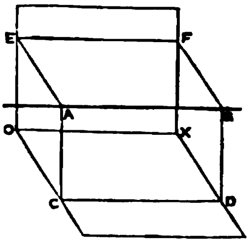
VERBAL DESCRIPTION.	EIDOGRAPH.	ORTHOGRAPH.

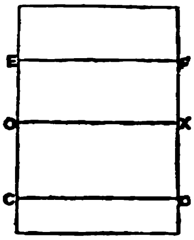
3. Cases of this are when the line is *in* one of the planes. In this case the line itself is its own projection upon that plane, and the axis is its projection upon the other.





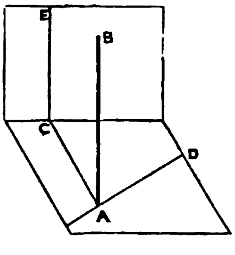
Class 3. Parallel to both planes, or to the axis.

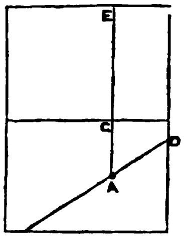




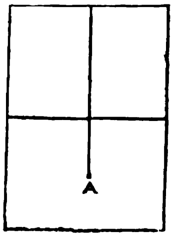
The only case of this class is as figured; and as the line meets no point in either plane, its traces cannot be marked. Its projecting planes are parallel also to the coordinate planes; and since they form with the axis, the sections of a plane by parallel planes, these projections are also parallel to the axis. Their distances from the axis on each plane are the same as the distances of the line from the other plane.

Class 4. Perpendicular to one plane, and therefore parallel to the other.





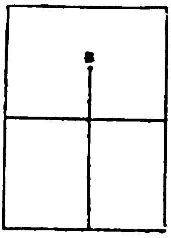
Parallel to Elevation.



In this case the projection on the plane to which the line is parallel will be perpendicular to the axis; and as the projection on the other plane is indefinite in position (all planes through the line being perpendicular to the plane), *any line whatever* through the point of section will define the trace, when taken in conjunction with AC.

It is, however, not necessary to trace the line AD at all, if we terminate CA at A, the point in which it meets the plane. The two orthographs will hence be simply as marked in the margin. This, because the simplest method, is that most commonly adopted in practice.

Parallel to Plan.



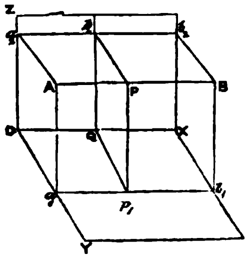
THE ORTHOGRAPH REPRESENTATION OF A POINT.

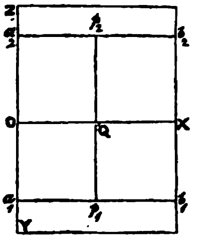
A point may be considered in three different ways: as the intersection of three planes; as the intersection of a plane and line; or as the intersection of two lines which are in the same plane. For special purposes, one of these views may be preferable to the others; but as appertaining to the general theory, they all coalesce under the second form in a very convenient state for practical use.

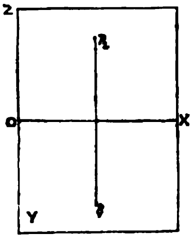
As in the line, so with regard to the point, we may define it by means of two of the intersecting figures that pass through the point; and we are at liberty to choose those positions which will most simply express those figures in reference to the coordinate planes. We shall, however, better perceive the relative values of the different special modes of representation by examining their results in succession. It may be advantageous to the student, however, to defer the consideration of the more complex expressions of a point in orthograph till he has performed a few of the preliminary problems; and we shall therefore merely lay before him in this place the mode of definition which has been found most convenient, and which is consequently most usually employed.

The method is this :—

Take three planes, two of them parallel to the coordinate planes, and the third perpendicular to them both. The projections of the line formed by the first two are parallel to the axis, and the intersections of the third with both planes will be perpendicular to the axis and pass through the same point.







In the first of these figures suppose P to be the point; then a_1b_1, a_2b_2 are the projections of a line parallel to the axis, which passes through the point to be represented: and p_1Q, Qp_2 are the traces of a plane perpendicular to them both, which also passes through P. Then it is quite clear that p_1 is the projection of the point P of the line AB upon the *plan;* and that p_2 is that of the same point upon the *elevation.* Moreover, since a_2B and p_2p_1 are planes, both perpendicular to the elevation, it follows that Pp_2, their intersection, is perpendicular to the *elevation.* In like manner Pp_1 is perpendicular to the *plan.* Also, it is clear that p_1p_2 is a rectangle, or Pp_2 = p_1Q and Pp_1 = p_2Q; whilst Pp_1, Pp_2 are the distances of the point P from the plan and elevation respectively.

We pass from this to the first immediate orthograph; in which we see that the point is defined by two horizontal lines, one on each coordinate plane, and by a line perpendicular to them both. For in revolving to the plan the elevation p_2Q is always at right angles to OX, and hence in orthograph continuous of p_1Q.

But as these horizontal lines a_1b_1, a_2b_2 serve no other purpose in definition than to mark the distance of the point from each plane, we may, knowing those distances, define the point as in the second orthograph by terminating the perpendicular through Q at those distances respectively. This is the most simple mode of all.

The points p_1, p_2 are called the *projections* of the point P; and the lines Pp_1, Pp_2 in the eidograph are the *projecting lines.* They are defined as the perpendiculars from the point to the coordinate planes in other works on this subject.

NOTATION.

This subject is one of scarce less importance in Descriptive Geometry than it is in algebra and its applications; and much attention has been given to it by the French writers on the subject, and more especially with respect to *graphic* exhibition. We shall follow them strictly in this respect, but in the literal notation we shall employ a method which is more symmetrical at the same time that it is more discriminative than theirs.

1. *Graphic Notation.*

1. The *data* and *quæsita* of a problem are always marked in continued line, when they are upon the visible parts of the plan and elevation; and when behind the elevation or below the plan, so as to be concealed by these planes, they are marked in round dots; thus, ————————, and . . . .

2. Lines used in construction, as the intersections of the projecting planes with the coordinate planes, etc. (except they belong to the preceding class), are marked in traits, whether visible or hidden, by the coordinate planes; thus – – – – – –.

3. When of the constructive lines some are very important in relation to the problem, they are marked in a mixture of points and traits; thus,

— · — · — · — · , or — · · — · · — · · —.

The number of interposed dots implying a greater degree of relative importance.

4. To distinguish the traces of a plane from the projections of a line,

in all the cases, a slight cross trait is placed at or near the end of a trace; thus,

————— |, or . . . . |, or .—.—.—.|, or —..—..|

In actual drawings much simplicity would be effected by denoting the traces, projections, and auxiliary lines by different colours. No fixed rule for this purpose has, however, been adopted.

2. *Literal Notation.*

1. In the eidograph, a point is represented by a Roman capital, and its projection on the plan and elevation respectively by the small letter with a subscribed numeral 1, 2. Thus a point A in space referred by projection to the plan and elevation will be denoted by a_1, a_2. If we represent it on the profile, we shall denote it by a_3.

2. A plane, being defined by its traces, will be denoted by Roman capitals, where it cuts the axis, the other literal denomination on the plan being marked with the same letter having 1 subscribed, and on the elevation by 2. Thus a plane whose traces meet the axis will be expressed by S_1SS_2, S_1S being on the plan and SS_2 on the elevation.

3. The traces of a line on the coordinate planes is represented by the Greek letters α, β, γ, δ, etc., with subscribed numerals. Thus the line $\alpha\beta$ will signify generally a line which meets the planes in α and β; and as α is on the plan, it will be marked α_1; and β on the elevation, it will be marked β_2. The projections of these points in the axis will therefore, in conformity with the principle, be α_2 and β_1.

4. The projecting planes of a line will bear that name which has two of its points of the same name with numerals subscribed. Thus $\beta_2\beta_1\alpha_1$ is called the plane β, and $\alpha_1\alpha_2\beta_2$ is called the plane α.

Or, again, by the letter at which the projecting plane cuts the axis, which is more simple, and is that to be used in the work.

SECTION II.

PRELIMINARY PROPOSITIONS.

The following are the problems referred to at p. 140, as being required in Descriptive Geometry.

The postulate assumed here, besides the postulates and propositions of plane geometry, is:—

That through three points not in one line, or through one point and a line, or through two lines which are parallel, a plane may be drawn or constituted.

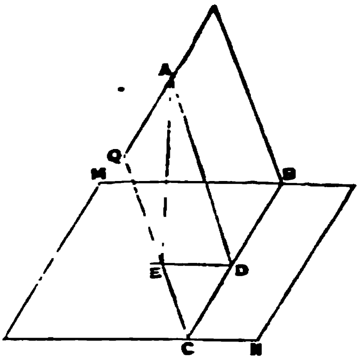
In any plane so drawn, of course all the operations of plane geometry may be performed.

PROPOSITION I.

To draw a perpendicular to a given plane from a given point.

1. Let the given point A be without the given plane MN.

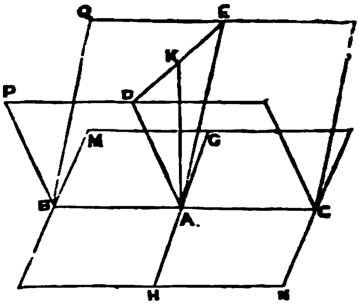
Through A draw any plane QB cutting MN in BC; and in the plane QB draw the line AD perpendicular to BC; in the plane MN, the line DE also perpendicular to BC; and in the plane ADE the line AE perpendicular to DE. This is the perpendicular required.*



For since BC is perpendicular both to AD and DE, it is perpendicular to the plane ADE (*Pls.* II. 2); and hence the plane MN through BC is perpendicular to the plane ADE (*Pls.* II. 16).

Wherefore AE is drawn in one of two perpendicular planes ADE and MN, at right angles to the common section DE; and is, therefore, perpendicular to MN (*Pls. Def.* 4.).

2. Let the given point A be in the given plane MN.



Through A draw any line BC in MN, and through BC any two planes PC, QC; in these planes draw the lines AD, AE perpendicular to BC; through AD, AE draw a plane cutting MN in GH; and in this plane draw AK perpendicular to GH. This will be the perpendicular required.

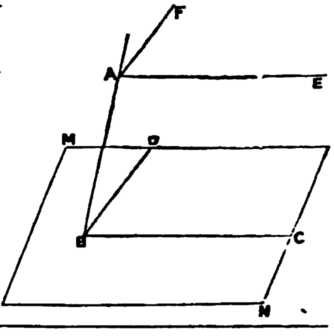
For, as before, AD, AE being perpendicular to BC, the planes DAE, MN are perpendicular; and AK in one of them DAE being perpendicular to the common section GH, is perpendicular to the other.

SCHOLIUM. The mode usually directed for the construction of this case is:—take any other point without MN, and by the preceding case draw a perpendicular to MN; and then through A draw AK parallel to this perpendicular.

This *seems* shorter: its brevity, however, is only in the statement, not in the work. Either method may be used as shall be most convenient in each particular case of its application.

PROPOSITION II.

Through a given point to draw a plane parallel to a given plane.



Let A be the given point, and MN the given plane.

Draw any line AB to meet MN in B; and in MN any two lines BC, BD; in the plane ABC draw AE parallel to BC, and in ABD draw AF parallel to BD. Then the plane through AE, AF is that required.

For the two lines CB, BD which meet in B being parallel to the two EA, AF which meet in A, they are in parallel planes (*Pls.* I. 7).

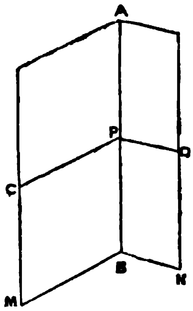
---

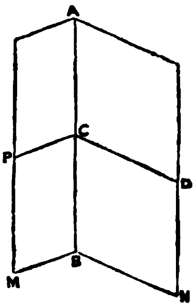
* If it should happen that ADE is a right angle, the construction will terminate with drawing AD; but this can only happen accidentally.

PROPOSITION III.

Through a given point to draw a plane perpendicular to a given line.

1. Let AB be the given line, and P the given point situated in AB.





Through AB draw any two planes AM, AN; and in these PC, PD perpendicular to AB. The plane through CP, PD is that required (*Pls. Def.* 5).

2. Let AB be the given line, and P the given point without it.

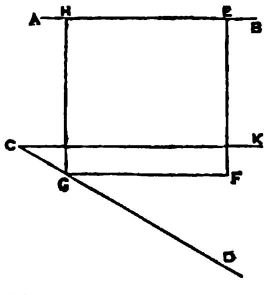
Through AB draw two planes AM, AN, the former of which passes through P and the latter any whatever; in the plane AM draw PC perpendicular to AB meeting it in C; and in the plane AN draw CD perpendicular to AB. The plane through PC, CD is that required.

PROPOSITION IV.

To draw a straight line which shall be perpendicular to two given lines.

Let AB, CD be the given lines.

Through any point C in CD draw CK parallel to AB; and from any point E in AB, a line EF perpendicular to the plane KCD; draw FG in this plane parallel to CK meeting CD in G; and in the plane EFG draw GH parallel to EF. This line will meet AB, and be perpendicular to both the lines AB and CD.



For since AB and FG are both parallel to CK, they are parallel to one another (*Pls.* I. 6); and hence in the same plane. Wherefore also EF is in that plane; and GH, being parallel to EF, is also in that plane, and consequently meets the line AB.

Again, because EF is perpendicular to the plane KCD, and GH parallel to EF, the line GH is also perpendicular to the plane KCD (*Pls.* II. 14); and hence it is at right angles to the line CD, which passes through G in that plane.

Lastly, since EF is perpendicular to the plane KCD, the angle EFG is a right angle; and since AB is parallel to GF, and HG to

EF, and one of the angles EFG a right angle, the figure HF, is a rectangle. The angle GHE is hence a right angle.

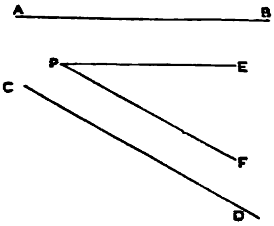
The line GH is therefore perpendicular to both the given lines AB, CD as was required.

PROPOSITION V.

Through a given point to draw a plane parallel to two given lines.

Let AB, CD be the given lines, and P the given point.

Through P draw PE, PF respectively parallel to AB, CD: then the plane through these is the one required.



For, since AB is parallel to a line PE in the plane EPF, it is parallel to that plane (*Pls.* I. 3); and similarly, CD is also parallel to that plane. Or, which is the same thing, the plane EPF is parallel to the given lines AB, CD.

PROPOSITION VI.

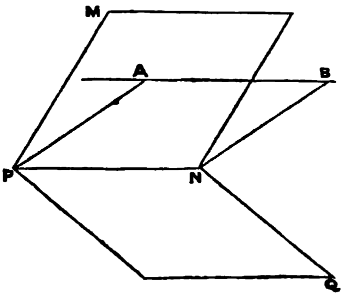
Through a given point to draw a line parallel to two given planes.

Let A be the given point, and MN, PQ the given planes.

Let PN be the common section of the given planes; through A and PN draw the plane APNB; and in this plane draw AB parallel to PN.

Then AB is the line required.

The truth of this follows from (*Pls.* I. 9).

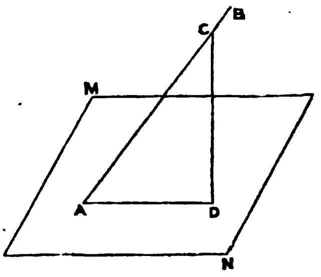


PROPOSITION VII.

Through a given line to draw a plane perpendicular to a given plane.

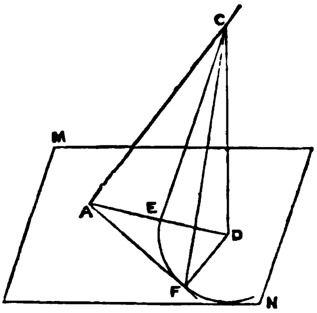
Let AB be the given line, meeting the given plane MN in A; to draw a plane through AB perpendicular to MN.

Take any point C in AB, and draw CD perpendicular to MN: then the plane through AB, CD is that required.



PROPOSITION VIII.

Through a given line to draw a plane to make a given angle with a given plane.

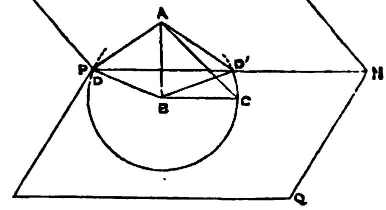


Let AC be the given line, meeting the given plane in A, and CD a perpendicular to this plane. Join AD, and make the angle CED equal to the profile of the given angle; describe the circle EF with D as centre and DE as radius; and draw the tangent AF to it: then the plane CAF is that required.

For join CF, FD. Then the two triangles CED, CFD are right angled at D, and have the sides about the equal angles common or equal. Whence CFD = CED. But CFD is the profile angle of the planes MN and ACF (*Pls.* II. 13, *and Def.* 6); and is hence equal to the given profile angle.

PROPOSITION IX.

Through a given point in one of two oblique planes to draw a line which shall make a given angle with the other plane.



Let A be a given point in one MN of two oblique planes; it is required to draw a line from A in MN which shall make a given angle with the other plane PQ.

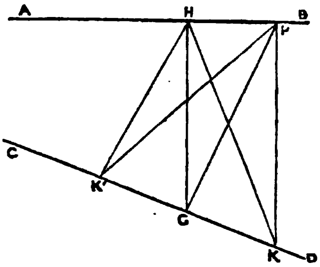
Draw AB perpendicular to PQ, and BC in any direction in the plane PQ; make the angle ACB equal to the given angle; with B as centre and BC as radius, describe the circle DCD′, cutting PN (the intersection of the planes) in D and D′. Then AD, AD′ both fulfil the conditions.

The proof is obvious and immediate.

PROPOSITION X.

Two straight lines in space being given, and a point in one of them: it is required to draw through the given point a line to meet the other given line, and make equal angles with them both.

Let AB, CD be the given lines, and P the given point in AB; it is required to draw from P a line to make equal angles with AB and CD.



Find GH perpendicular to AB, CD (*Prop.* IV.); make GK and GK' each equal to HP; and join PK, PK'. Each of these lines is equally inclined to AB and CD.

For, since HG, HP are equal to HG, GK, and the included angles GHP, PGK are right angles (*Constr.*), the base HK is equal to the base PG.

Also, since the two sides HP, HK are equal to the two KG, GP, and the base KP common, the angles KPH, GKP are equal. And these are the angles stated in the enunciation to be equal.

In the same manner it may be shown that KP makes equal angles with AB and CD.

PROPOSITION XI.

To construct a trihedral angle equal to a given one.

This problem branches out into several cases according to the essential data by which the pattern trihedral is given. In all cases, however, it is possible, in virtue of the theorems already given, to completely construct the trihedral angle or its supplementary one from adequate data.

Suppose this done, we may copy this trihedral angle by taking the three most convenient of its six parts (three plane angles and three dihedral angles) as one data.

The student should be required to discuss the several cases in order.

SECTION III.

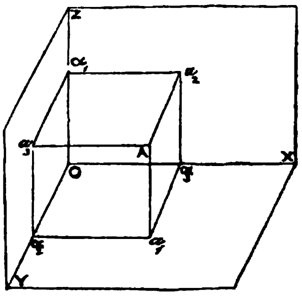
CONSTRUCTIONS RELATING TO THE PLANE, LINE, AND POINT.

PROPOSITION I.

The projections of a point on two of the coordinate planes are given to find its projection on the third.

Let A be the given point, and YX, XZ, ZY the three planes of projection; there are given a_1, a_2 the projections of A on the planes XY, XZ to find a_3 on YZ.

Eid.



Now in the eidograph, the three lines Aa_1, Aa_2, Aa_3 are perpendicular to the three planes; and hence the planes through them two and two are parallel to the coordinate planes and the parallel edges are equal, that is,

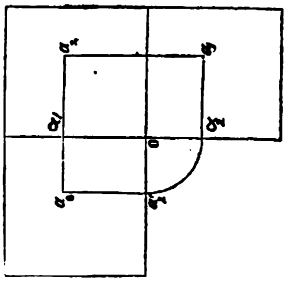
$$a_3\alpha_2 = O\alpha_1 = a_2\alpha_3 = Aa_1,$$
$$a_3\alpha_1 = O\alpha_2 = a_1\alpha_3 = Aa_2,$$
$$a_2\alpha_1 = O\alpha_3 = a_1\alpha_2 = Aa_3.$$

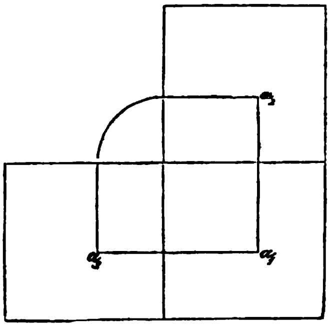
Now if we make the plane ZY revolve about OZ till it coincides with XZ, behind OZ; then the line OY will coincide with OX, and a_3a_1 with $a_1a'_3$ the points a_2, a_3 describing quadrants during the revolution.

Whence from O in the orthograph as centre, describe the quadrant $a_2a'_2$ and draw $a'_2a'_3$ perpendicular to OX, meeting the line a_1a_2 in a'_3: this is the orthograph with respect to the elevation and profile.

If with YZ and XY, we proceed in a similar manner, we shall get the orthograph of the point in respect of the plan and profile.

Orth. Elevation and Profile.



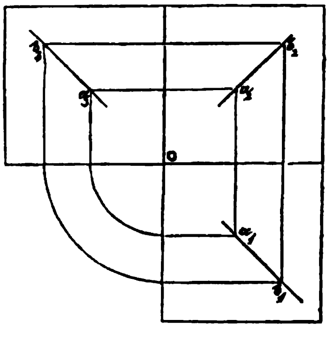


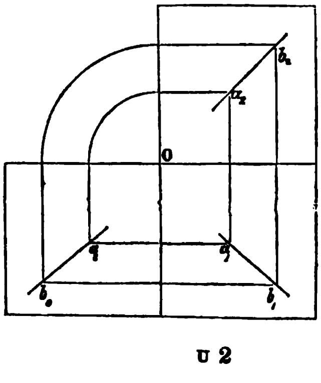
Orth. Plan and Profile.

PROPOSITION II.

The orthograph of a line on two planes is given to construct the orthograph of either of them with the third plane.

It is evident that since a line is given when two points in it are given, that if we take the projections of any two points in it, and employ the preceding proposition, we shall only have to draw the line through the two projections upon the third plane to resolve the problem.

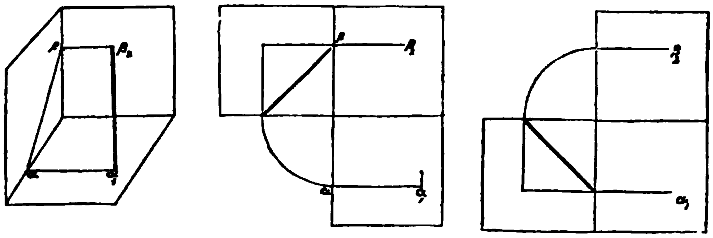




These orthographs are representative of the elevation and section, and of the plan and section respectively.

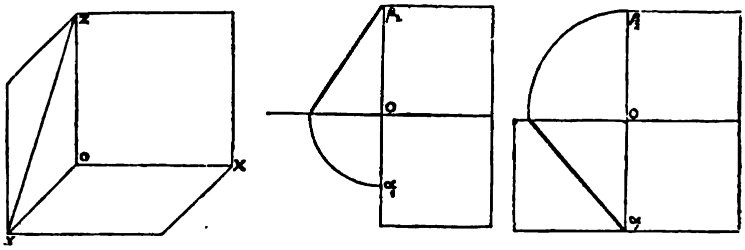
Scholium. 1. When the line is given under some particular circumstances, the construction is simplified. Three cases deserve notice from their frequent occurrence in practice.

1. When the *traces* of the line are given.



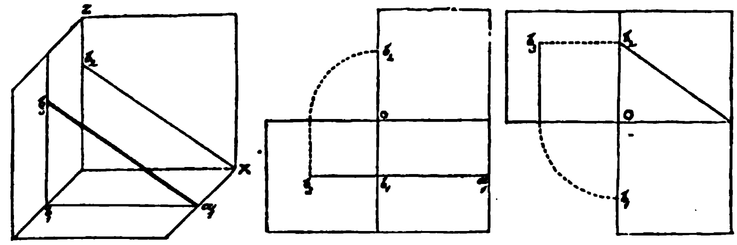
Let α_1, β_2 be the traces: draw as in the figure: then the dark line is evidently the projection on the third plane.

2. When the line itself is the section of any plane whatever with a profile plane; and that profile plane can be taken as the coordinate profile plane.



This is obviously only the particular case of the preceding, in which α_1 coincides with α and β_2 with β.

3. When the line is parallel to one of the coordinate planes.

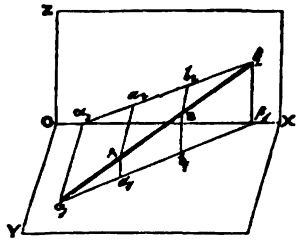


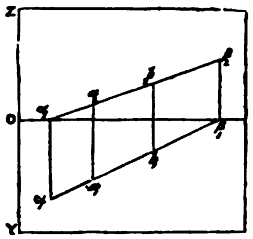
Scholium 2. It will be seen hereafter that the same process can be applied to any figures whatever. In fact it will be obvious that we can always construct by points as we have here done the line $\alpha\beta$.

PROPOSITION III.

Given the traces of a line to find its projections; and, conversely, given the projections of a line to find its traces.

(1) Let α_1, β_2 be the traces of the line, and through $\alpha_1\beta_2$ draw planes $\alpha_1\beta_1\beta_2$, $\alpha_1\alpha_2\beta_2$ perpendicular to the planes XY, XZ, cutting the two planes as in the figure of the eidograph.





Then, since XZ and $\alpha_1\beta_1\beta_2$ are perpendicular to XY, the line $\beta_1\beta_2$ is perpendicular to XY, and to the line OX in it. (*Pls.* II. 18). In like manner $\alpha_1\alpha_2$ is perpendicular to OX. Also, the projection on the horizontal plane is $\alpha_1\beta_1$ and that on the vertical is $\beta_2\alpha_2$. Wherefore these projections can be found without actually describing the projecting planes; since from the positions of α_1, β_2 and the axis OX, we can find β_1 and α_2.

Again, had the vertical plane been turned to the horizontal as in the orthograph, and the points α_1, β_2 been given, the same processes might be performed upon the united plane that were separately performed upon the two planes in eidograph. Whence it follows, that if we:—

Draw $\beta_2\beta_1$, and $\alpha_1\alpha_2$ perpendicular to OX, and join $\alpha_1\beta_1$, $\alpha_2\beta_2$, these will respectively be the horizontal and vertical projections which were required.

(2) The converse follows from the same considerations, *viz.*:—

Draw the projections a_1b_1, a_2b_2 to meet the axis in β_1, α_2; and then perpendicular to the axis, draw $\alpha_1\alpha_2$ to meet a_1b_1 in α_1, and $\beta_1\beta_2$ to meet a_2b_2 in β_2: then α_1, β_2 are the traces of the line whose projections are a_1b_1, a_2b_2.

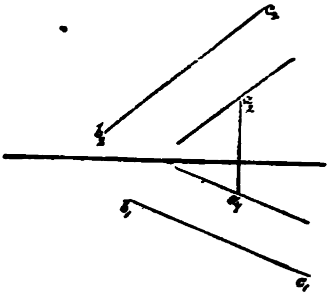
Scholium. In the preceding figure, the traces are given for the line lying in the first region; but the process is the same for all the others.

Cor. 1. If the projections of two points in the line be given, the same things may be found, viz., the traces and the projections of the line passing through them: for the projections of the line will pass through the projections of the points.

The projections being found by drawing these lines on the coordinate planes; then the traces of the lines will be found as in the problem itself.

Cor. 2. Hence, also, if a point and a line be given by means of their projections, the projections and traces of a line through the given point parallel to the given line can be found.

Thus, let a (or a_1a_2) be the given point, and bc (or b_1c_1, b_2c_2) the



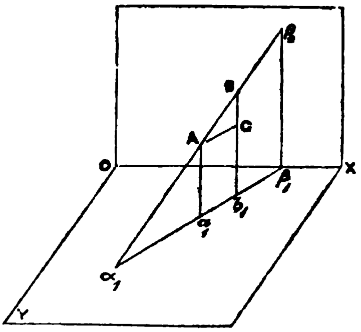
given line; then lines through a_1 and a_2 parallel to b_1c_1 and b_2c_2 will be the projections of a line through a parallel to bc.

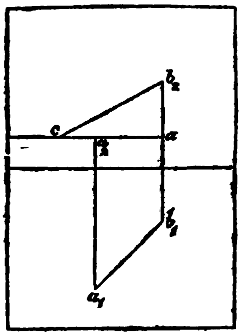
For, when lines are parallel their projections on any plane will be parallel, and the converse: but in the present case, the projections are, by construction, parallel; and hence they are the projections of parallel lines.

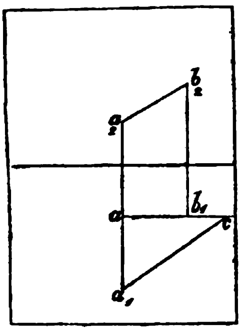
PROPOSITION IV.

Given the projections of two points in a line, to construct a line equal to the distance of those points.

In the eidograph, let AB be the segment of the line $\alpha_1\beta_2$, a line equal to which is to be constructed in orthograph, having given the projections a_1a_2, b_1b_2.







Let $\alpha_1\beta_1\beta_2$ be the plane which projects the line horizontally, and Bb_1, Aa_1 the lines which project the points A, B. Draw AC parallel to $\alpha_1\beta_1$ meeting Bb_1 in C.

Then since $\alpha_1\beta_1\beta_2$ is a right angle, and AC is parallel to $\alpha_1\beta_1$, ACB is a right angle, and therefore given. Also, since Aa_1 is parallel to

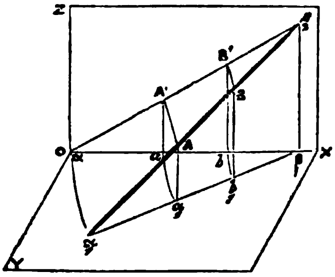
Bb_1 and AC to a_1b_1, we have AC = a_1b_1, and therefore given. And, finally Aa_1, Bb_1 are the heights of the projections on the vertical plane of A and B above the axis OX; and, therefore, likewise given, as well as the difference BC.

Wherefore, we have given the two sides AC, CB and the included right angle C; the hypothenuse AB of which triangle is the length required.

Whence, in the first orthograph, if we draw a_2a parallel to the axis, and take in it $ac = a_1b_1$, and join cb_2, this line is equal to AB: for its sides about the right angle are equal respectively to those of the triangle ABC about its right angle.

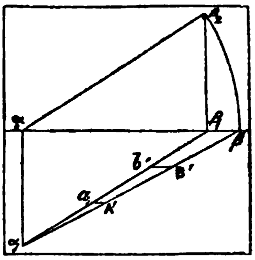
Cor. 1. Had we employed the plane which projects the line upon the vertical coordinate plane, the same reasoning would have led to the construction indicated by the second orthograph.

Cor. 2. When the traces of the line itself are given, as well as the projections of the points upon *one* of the planes are given, the following method will be convenient.



Conceive the projecting plane for the horizontal projection to revolve about $\beta_1\beta_2$ till it coincides with the vertical coordinate plane XZ: then the points α_1, a_1, b_1 will describe circular arcs $\alpha_1\alpha$, a_1a, b_1b; and the lines $\alpha_1\beta_2$, Aa_1, Bb_1 will take the positions $\alpha\beta_2$, A$'a$, B$'b$; and AB will coincide with A$'$B$'$. Whence, in the orthograph we have the rule:—

With centre β_1 describe (the figure is easily conceived) the arcs b_1b, a_1a, $\alpha_1\alpha$; join $\alpha\beta_2$; and draw aA$'$, bB$'$ parallel to $\beta_1\beta_2$: then A$'$B$'$ is the length required.



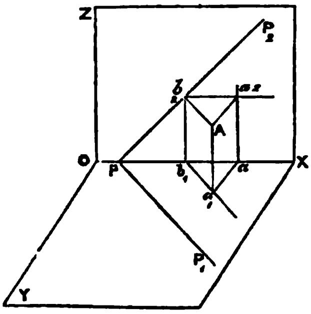
Cor. 3. The following method, adapted to the case of *Cor.* 2, is rather simpler as to actual work: the deduction of it from the eidograph is left as an exercise for the reader.

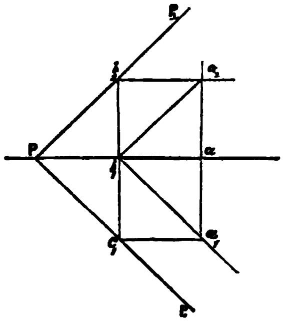
From centre α_2 with radius $\alpha_2\beta_2$ describe the arc $\beta_2\beta$ to cut the axis in β; join $\alpha_1\beta$; and draw b_1B$'$, a_1A$'$ parallel to the axis: then A$'$B$'$ is equal to the segment of the line required.

Scholium. The student should take into consideration the circumstances of this problem under special relations of the data, as several such occur in subsequent problems.

PROPOSITION V.

Given one of the projections of a point situated in a given plane to find the other.





In the eidograph, let the horizontal projection a_1 of the point A situated in the plane P be given. Through A let a line Ab_2 be drawn in the plane P parallel to its horizontal trace PP_1, and let the projecting planes of this line be drawn.

Then since Ab_2 is parallel to PP_1 the section of the planes P and XY, it is parallel to XY, (*Pls.* I. 3): wherefore since the point a_1 is given, the line a_1b_1 through it parallel to the given line PP_1 is also given.

Again, the line Ab_2 being in the plane P, has its trace b_2 in the trace PP_2 of the plane; and the line b_1b_2 is perpendicular to the axis. Whence, having the point b_1 and the trace PP_2 given, the point b_2 is given.

Finally, the projection of Ab_2 on the vertical plane is parallel to the axis, since Ab_2 is parallel to XY, and its projecting plane is evidently parallel to the axis, and the point b_2 in it is given; whence the line b_2a_2, the projection of Ab_2, is given; and its intersection with aa_2 is given. That is, the projection on the vertical plane is given.

Wherefore the orthographic construction.

Through the given horizontal projection a_1 draw a line parallel to the horizontal trace PP_1 meeting the axis in b_1; draw b_1b_2 perpendicular to the axis, meeting the vertical trace in b_2; draw b_2a_2 parallel to the axis; and finally a_1a perpendicular to the axis meeting b_2a_2 in a_2. This is the vertical projection sought.

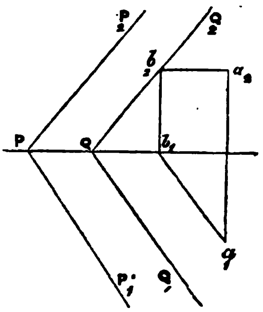
We might have operated in a similar manner if the vertical projection a_2 had been given instead of the horizontal one a_1.

COR. If the projections of a point and the traces of a plane be given, *either* of the constructions (the horizontal or the vertical) will furnish a criterion, *whether a given point be in a given plane or not*—a criterion that in practice is often required. One alone of them is sufficient; for if one be fulfilled, the other will *necessarily* be so too.

PROPOSITION VI.

Through a given point to trace a plane parallel to a given plane.

Since the traces of parallel planes upon each coordinate plane are parallel, it is manifestly sufficient that we find one point in either of



the traces of the required plane to be able to construct it.

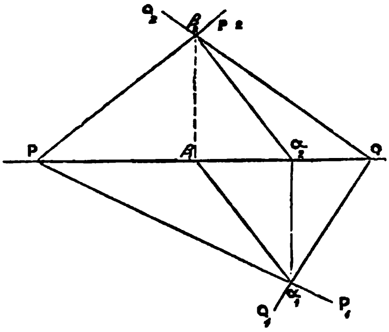
Draw a_1b_1 from the given horizontal projection a_1 of the given point parallel to the horizontal trace PP_1 of the given plane; draw a_2b_2 parallel and b_1b_2 perpendicular to the axis meeting in b_2: then b_2 is a point in the vertical trace of the plane required.

Draw QQ_2 through b_2 parallel to PP_2 meeting the axis in Q; and through Q draw QQ_1 parallel to PP_1: then Q_1QQ_2 is the plane required.

PROPOSITION VII.

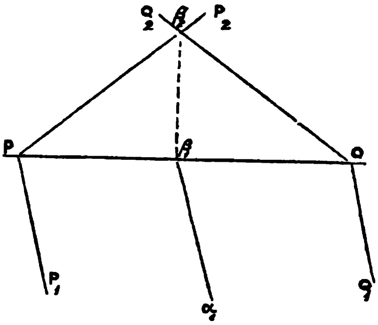
Given the traces of two planes, to find the projections of their line of intersection.

This problem has several cases, which, on account of its great practical importance, we shall bring under consideration successively.



Case 1. Let the traces of the planes P and Q upon both coordinate planes meet within the limits of the drawing.

Then their intersections α_1, β_2 upon the horizontal and vertical planes will be the respective traces of the line of intersection; and the projections may be constructed by *Prop.* III. as indicated in the figure.

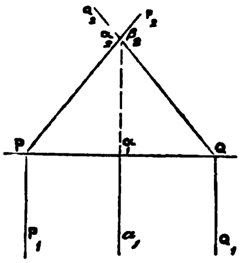


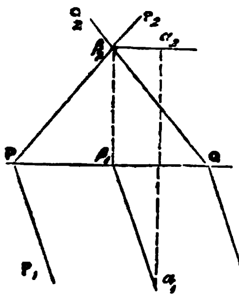
Case 2. Let the traces of the planes upon one coordinate plane intersect within the limits of the drawing, and the other traces be either parallel to one another or meet beyond the drawing.

Draw $\beta_2\beta_1$ perpendicular to the axis: this will obviously be one point in the horizontal projection (adapted to this figure): and draw $\beta_1\alpha_1$ *tending to* the intersection of PP_1, QQ_1 if they would meet, or parallel to them if they be parallel. This will be the horizontal projection of the line of intersection of the planes P and Q.

To find the projection on the vertical plane, we remark:—

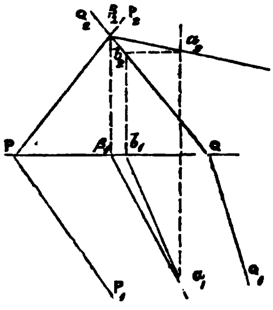
(1.) If PP_1, QQ_1 be parallel, the section of two planes through them will be parallel to the plane in which they are situated. Whence, if PP_1, QQ_1 be perpendicular to the axis, the vertical projection of the line of section will be a single point; and if they be not, the vertical projection will be parallel to the axis. These two cases are thus orthographed :—





(2.) When the horizontal traces of P and Q would meet, but beyond the limits of the drawing,—

Take any point a_1 in the horizontal trace, and draw a_1a_2 perpendicular to the axis; draw a_1b_1 parallel to the trace QQ_1 and b_1b_2 perpendicular to the axis, meeting the trace QQ_2 in b_2; and b_2a_2 parallel to the axis, meeting a_1a_2 in a_2: then β_2a_2 is the vertical projection of the line of section of P and Q.



For it follows, from what was shown in *Prop.* v., that a_1a_2 is a point in the plane Q: and since a_1 is the horizontal projection of a point in the line of section of the two planes, a_2 is the vertical projection of that point. Whence a_2, β_2 being the vertical projections of two points in the line, $a_2\beta_2$ is the vertical projection of the line itself.

CASE 3. Let both the pairs of traces intersect beyond the limits of the drawing.

Cut them both by any convenient third plane, and find the projections of the two lines of its intersection with them. The two horizontal ones of these projections will intersect in the horizontal projection of the point in which the third plane would meet the line of section sought, and the vertical projections in the vertical projection of the same point. We thus have the projections of one point in the line required.

If we find in a similar manner the projections of a second point by means of a second subsidiary plane, we shall have the requisite data for drawing the projections of the line sought.

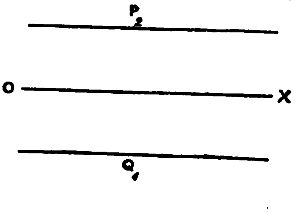
CASE 4. Let the planes be parallel.

Here there will be no intersection at all; since there is no line of section in fact, and consequently no line in orthorgraph.

CASE 5. Let the two planes be perpendicular to the coordinate planes, one to one, and the other to the other.

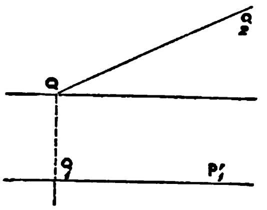
Then their traces on these respectively are the projections of the line of section upon the planes of projection to which the given planes are perpendicular. There are several varieties which, though simple, are of frequent occurrence.

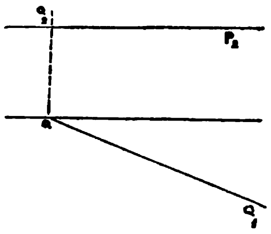
(*a.*) The general case has been already figured. (See *Prop.* I.)



(*b.*) When both projecting planes are parallel to the coordinate planes, both projections are parallel to the axis.

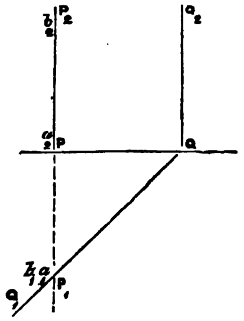
(*c.*) Let one only of the planes be parallel to a coordinate plane: then according as the said plane is parallel to the vertical or horizontal plane, the figures are

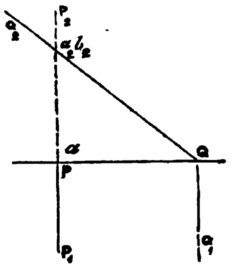




where P is in both cases the plane which is parallel to the coordinate plane.

(*d.*) Let one of the planes P be perpendicular to the axis, and the other Q be perpendicular to a coordinate plane: then according as Q is perpendicular to the horizontal or vertical coordinate plane, the figures will be;

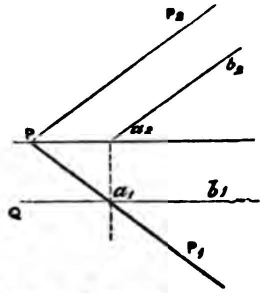


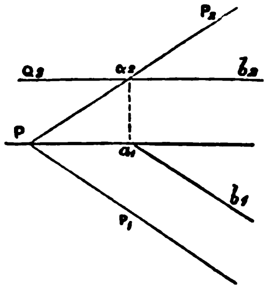


where b_1, a_1 coalesce in the first case, or the projection is a single point; and a_2b_2 in the second case, the projection being also a single point. The other projections are an indefinite straight line perpendicular to the axis.

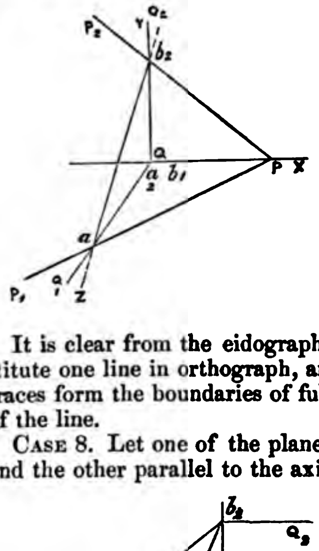
(*e.*) If both planes should be perpendicular to the axis, they will either be parallel or coalesce. On the former hypothesis there is no intersection; and in the latter every line in the coalescent planes may be considered an intersection, and the problem admits of innumerable answers, or is indeterminate.

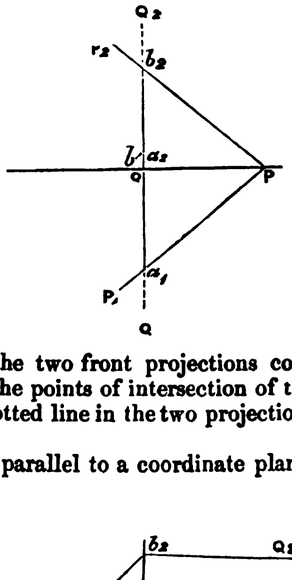
CASE 6. Let one plane be parallel to a coordinate plane, and the other cut the axis. An inspection of the annexed figures will be a sufficient indication to the attentive student.





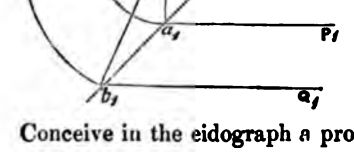
Case 7. Let one plane be a profile plane, and the other oblique to the axis.

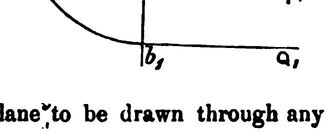




It is clear from the eidograph, that the two front projections constitute one line in orthograph, and that the points of intersection of the traces form the boundaries of full and dotted line in the two projections of the line.

Case 8. Let one of the planes P, be parallel to a coordinate plane, and the other parallel to the axis.





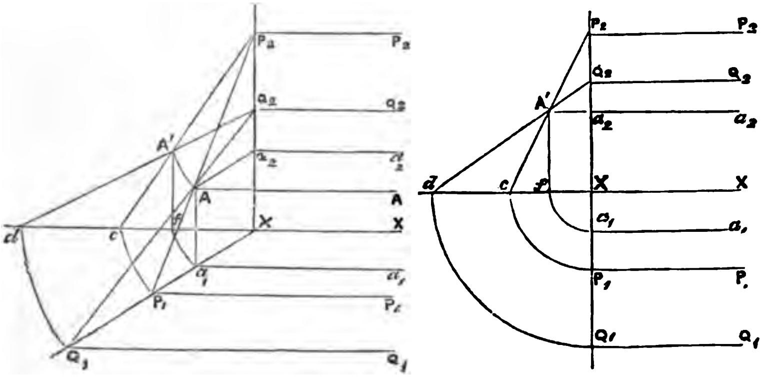
Conceive in the eidograph a profile plane to be drawn through any point A in the line of intersection. It will cut the coordinate planes in two lines b_1A', $A'b_2$ at right angles to one another; the plane Q in the line b_1b_2; and the plane P in the line Aa_1. Draw Aa_2 parallel to a_1A', and then from a_2 the line a_2a' parallel to the axis.

Now, since Aa_2 is parallel to $A'b_1$, it is parallel to the horizontal coordinate plane; and hence a_2 is the height of the projection of A above the axis.

Conceive now the profile plane, with the system of lines and points in it, to be revolved into coalescence with the vertical coordinate plane. The points a_1, b_1 will describe circular arcs; the point a_1 will be at c, b_1 at d, the line a_1A will coincide with cc' and the point A with c', and c' will be in the line a_2a'.

Suppose now that in orthograph, the same letters as far as they are common to represent the same lines when the vertical plane coincides with the horizontal. Then describe the arcs a_1c, b_1d; join db_2; draw cc' perpendicular to the axis, and $c'a_2$ parallel. Then a_2a' and a_1P_1 are the projections sought.

CASE 9. Let both planes be parallel to the axis.



Conceive the eidograph, as in the preceding case, to be cut by a profile plane: the coordinate planes in Q_1X, XQ_2; the planes P and Q in P_1P_2 and Q_1Q_2; and their line of section in A. Then the line A is parallel to the axis, and its projections are therefore also parallel to it. Also the lines Aa_1, Aa_2 parallel to XQ_2, Q_1X meet the coordinate planes in the projections of the point A. Let now the profile plane be revolved till it coalesces with the vertical. Then $A'f$ is the height of the vertical projection of A above the axis; and Xf the distance of the horizontal projection of A in front of the axis; and the projections themselves parallel to the axis.

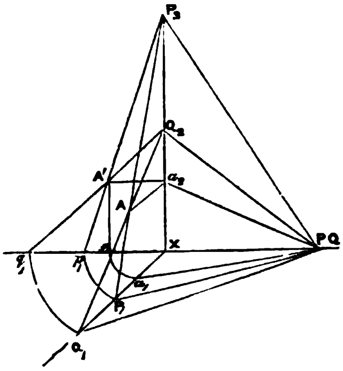
Hence, in orthograph, we have the following process:—

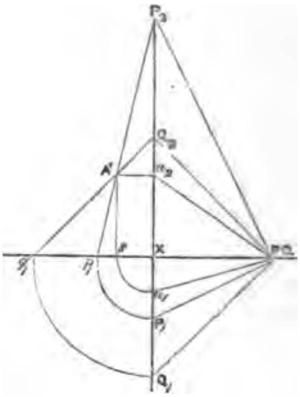
Draw a line perpendicular to the axis, cutting the traces of the planes, the first in P_1, P_2 and the second in Q_1, Q_2; set off XP_1, XQ_1 to c and d upon the axis; and join cP_2, dQ_2 intersecting in A': then the line through A' parallel to the axis is the vertical projection of the line.

Draw $A'f$ perpendicular to the axis, and set off Xa_1 equal to Xf: then a line through a_1 parallel to the axis is the horizontal projection of the line.

CASE 10. Let the two planes cut the axis in the same point.

In this case the line of intersection will pass through that point, and hence also its respective projections will also pass through it. One point in each projection is therefore known; and it will be sufficient to find that of one other point in that line.



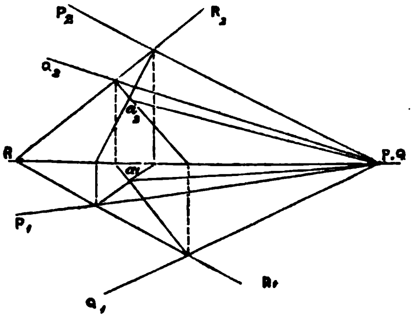


Conceive the system in eidograph cut by a profile plane, the section with P being P_1P_2 and with Q being Q_1Q_2; then these will intersect in A, a point in the line whose projections are required. Let the profile system coalesce with the vertical plane; then it will appear, as in the last case, that if $A'a_2$ be drawn parallel to the axis, a_2 will be the vertical projection of A; and hence a_2 PQ being drawn, it will be the vertical projection of the line required.

The horizontal projection may be found by drawing $A'f$ perpendicular to the axis, setting off $Xa_1 = Xf$, and drawing a_1 PQ.

SCHOLIUM. All the cases that can occur range themselves under some of these; and it will be obvious that the two last are in reality so related as that the ninth is the limiting case of the tenth—when the point PQ is infinitely distant. They are both constructible in a different manner: viz., by employing *any* plane instead of the profile. We may remark that in the case where the given planes would meet beyond the limits of the drawing, a second profile plane will be necessary, in the method already given, for finding the projections of a second point in the line. The same remark applies to the method of construction which follows.

Let P, Q be the planes as before; and trace any other plane R to cut them both; construct he lines of intersection of P, R and of Q, R, as

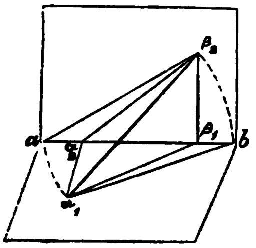


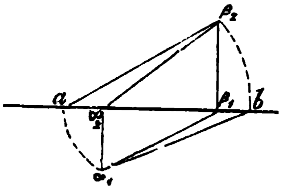
in the figure: then the intersections of these projections a_1, a_2 will be the intersections of a point in the line which it is required to project.

For it is obvious that as each of the planes PQ passes through the required line, their intersections with R will also be in that line; and the intersections of P with R and of Q with R will therefore both be projected in lines which will contain the projections a_1, a_2 of that point.

PROPOSITION VIII.

To construct the inclinations of a given line to the coordinate planes.





In the eidograph, let $\alpha_1\beta_1$ and $\alpha_2\beta_2$ be the horizontal and vertical projections of the given line $\alpha_1\beta_2$. Then about the perpendicular $\beta_1\beta_2$ let the projecting plane of the horizontal revolve till it coincides with the vertical coordinate plane. In this case α_1 will describe an arc $\alpha_1 a$, making $\beta_1 a = \beta_1\alpha_1$, and $\alpha_1\beta_2$ will take the position $a\beta_2$. But the angle $\beta_2\alpha_1\beta_1$ is the inclination of the line $\alpha_1\beta_2$ to the horizontal coordinate plane; and hence also $\beta_1 a\beta_2$ is that inclination.

Whence, in orthograph, if having formed the projecting system of the given line, we make $\beta_1 a = \beta_1\alpha_1$, and join $a\beta_2$ we have the angle $\beta_1 a\beta_2$ as that required.

In the same way, making $\alpha_2 b = \alpha_2\beta_2$, and joining $b\alpha_1$, we have the inclination of the line to the vertical coordinate plane represented by $\alpha_2 b\alpha_1$.

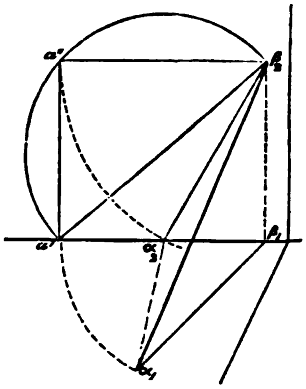
PROPOSITION IX.

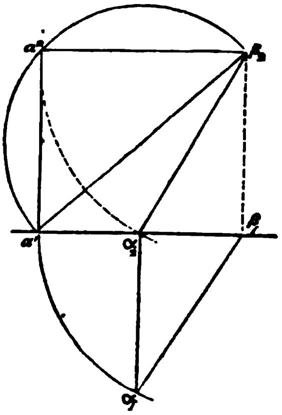
Given the angles which a straight line makes with the coordinate planes, and likewise a point through which it passes, to construct the line.

(1.) Let the given point, as β_2, be in the vertical coordinate plane, and let us conceive the problem executed in eidograph, $\alpha_1\beta_2$ being the line sought.

Let the projecting plane $\alpha_1\alpha_2\beta_2$ be revolved about $\alpha_1\beta_2$ till it coincides with $\alpha_1\beta_1\beta_2$, the other projecting plane; and let the system so transformed be revolved about $\beta_1\beta_2$ till it coincides with the vertical coordinate plane. The system will then be represented by the plane figure $a'a''\beta_2\beta_1$.

In this figure we have given the side $\beta_1\beta_2$ since β_2 is given on the vertical plane; the angle $\beta_2\beta_1 a'$ is a right angle, being the same with $\beta_2\beta_1\alpha_1$; and the angle $\beta_2 a'\beta_1$ is given since it is equal to the given inclination of $\alpha_1\beta_2$ to the horizontal coordinate plane. Whence $a'\beta_2$ is given.





Again, the angle a'' is given, being the right angle $\alpha_1\alpha_2\beta_2$, and a'' is in a given semicircle—that on $a'\beta_2$; the angle $a'\beta_2a''$ is given, since it is the angle $\alpha_2\beta_2\alpha_1$ made by the line with the vertical coordinate plane, and hence the position of the line $a''\beta_2$ is given. Whence the intersection of this line with the semicircle gives the length β_2a'', which is equal to $\beta_2\alpha_2$; and as β_2 is given, and α_2 is in the given axis, the point α_2 is given, and hence the perpendicular $\alpha_2\alpha_1$ to the axis, which contains the horizontal trace α_1 of the line is given.

But this trace is also in the circle described about β_1, with the radius β_1a', since $\beta_1a' = \beta_1\alpha_1$. Whence their intersection gives the horizontal trace of the line; and we have the following orthographic construction.

Draw $\beta_2\beta_1$ perpendicular to the axis, and make the angle $\beta_1a'\beta_2$ equal to the given angle which the line is to make with the horizontal plane: and on $a'\beta_2$ as diameter describe a semicircle, and at β_2 make the angle $a'\beta_2a''$ equal to the given angle which the line is to make with the vertical plane. From β_2 as centre, and β_2a'' as radius, describe a circle cutting the axis in α_2: draw the perpendicular $\alpha_2\alpha_1$, and describe a circle to meet it in α_1 from centre β_1 with radius β_1a'. Their intersection α_1 is the horizontal trace of the line.

(2.) When the given point is not in one of the coordinate planes, assume a point in one of them, and construct a line to fulfil the conditions as already directed: then through the projections of the given point draw parallels to those of the line found. These will define the line.

For this line is parallel to the former one; and parallel lines make equal angles with either plane.

PROPOSITION X.

A point and a plane are given to find:—

(1.) *The projections of a perpendicular to the plane from the point;*

(2.) *The length of the perpendicular;*

(3.) *The intersection of the perpendicular with the plane.*

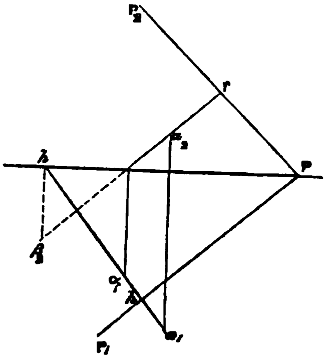
(1.) Conceive that two planes are drawn through the given point in the eidograph, perpendicular to the traces of the given plane; their sections with the given plane will be perpendicular to the given traces; their own traces on the coordinate planes will also be perpendicular to the traces of the given plane; their traces will also be the projections of their line of intersection with each other; and they will both be perpendicular to the given plane (*Pls.* II. 7, 18).

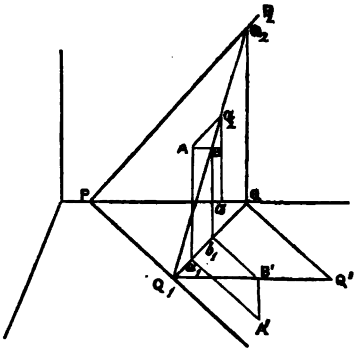
These two subsidiary planes passing through the given point, their intersection passes through it; and hence the projections of that line pass through those of the given point; and these projections being perpendicular to the given traces of the given plane, they are at once constructible.

Let P be the given plane, and a_1a_2 the given point: draw a_1h perpendicular to PP_1, and a_2v perpendicular to PP_2: these are the projections of the line required; and its traces, if sought, are found to be α_1, β_2 by the methods already explained.

(2.) Let P (second fig.) be cut by one of the two planes, Q, perpendicularly to its trace, supposed in the figure to be the horizontal one. This plane contains the perpendicular AB; and hence the foot of that perpendicular will be in Q_1Q_2 the intersection of P and Q, and AB will be at right angles to that line.

Now since A is given, a_1, its projection, is given in position, and aa_2 in magnitude (aa_2 being perpendicular to the axis); and the line QQ_1 through a_1 perpendicular to PP_1 is given, and the point Q its intersection with the axis; also since QQ_2 is perpendicular to the axis and meets a given line PP_2, the point Q_2 and the magnitude of QQ_2 are given.





Moreover, since $a_1A = aa_2$ is given in magnitude and position, and the point a_1 also given in position in the plane Q, we may turn that plane on the horizontal. In this case Q_2 takes the position Q'; A, B and Q_1Q_2 take the positions A', B', and Q_1Q'. We have then the positions and magnitudes of the lines Q_1Q', QQ', and $A'a_1$, to find the perpendicular A'B'. Whence the construction in orthograph.

Draw QQ_1 through a_2 perpendicular to PP_1, and QQ_2 perpendicular to the axis (the fig. in orthograph is left for the student to sketch): draw QQ', a_1A' parallel to PP_1, and make $QQ' = QQ_2$ and $a_1A' = aa_2$, and join Q_1Q': and finally draw $A'B'$ perpendicular to Q_1Q'. It will be the perpendicular required.

(3.) Draw $B'b_1$ parallel to $A'a_1$, and b_1b_2 perpendicular to the axis, meeting the perpendicular from a_2 to PP_2. Then b_1b_2 will be the projections of the point of intersection; as will be manifest from a comparison of the orthograph with the eidograph.

SCHOLIUM. 1. When, as is often the case, the length of the perpendicular is alone required, the method here given is more rapidly executed than that given by other authors; and when both the intersection and length are required, it has still some advantage. The quæsita are found, however, in the reverse order here from the usual process.

The ordinary method is still very elegant; and it applies to finding the intersection of *any* given line with a given plane. In that general form it will be given: and when the point of section is found by that process, the length of the perpendicular is found as the distance of that point from the given point, by a preceding proposition.

SCHOLIUM. 2. The converse problem is soluble in the same way, viz.:—

Through a given point to trace a plane perpendicular to a given line.

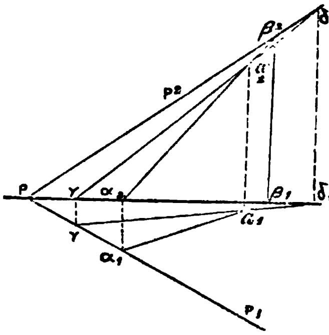
Let $\alpha_1\beta_2$ be the given line, and a_1a_2 the given point.

Now, by the property established in the proposition itself, any plane P, whose traces are perpendicular respectively to $\alpha_1\beta_1$ and $\alpha_2\beta_2$ will be perpendicular to $\alpha_1\beta_2$. Let such plane be traced: then (*Prop.* VI.) through a_1a_2 trace a plane Q parallel to P: it will be that required.

PROPOSITION XI.

Given the projections of a line and the traces of a plane, to find the projections of their intersection.

It is obvious that if we draw any two planes through the given line to cut the given plane in two lines, the intersections of these lines in the given plane will be the same point as the intersection of the given line and given plane. Whence, also, the intersection of the projections of these two lines upon the coordinate planes will be identical with the projections of the intersection of the line and plane.



When the line is given by means of its projections, the two projecting planes themselves will be most conveniently employed as these two subsidiary planes. Also, when the traces of the given line define it, this process is on the whole the more convenient; but when the line is given as the intersection of two given planes, as is not unfrequently the case, these planes themselves are far more convenient subsidiary planes than any other. The *figure* here given is constructed by means of the projecting planes.

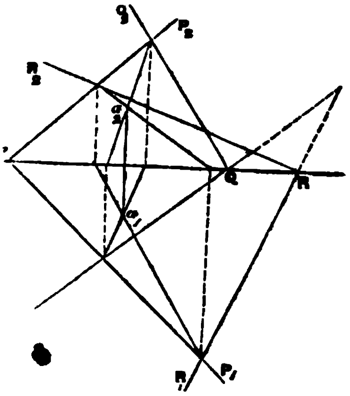
Let $\alpha_1\beta_1\beta_2$ be the projecting plane on' the horizontal coordinate plane: the projections of the intersection of this plane with the given plane P are $\alpha_1\beta_1$ and $\alpha_2\beta_2$. Similarly if $\gamma_2\delta_2\delta_1$ be the projecting plane on the vertical coordinate plane, the projections of the intersection of this plane with the given plane P are $\gamma_1\delta_1$, $\gamma_2\delta_2$ γ_2 is on the axis and γ_1 on PP_1. The intersection of $\alpha_1\beta_1$ and $\gamma_1\delta_1$ gives a_1, the horizontal projection of the point; and that of $\alpha_2\beta_2$, $\gamma_2\delta_2$ gives a_2, the vertical projection of the point.

As a criterion of accuracy of work, it may be remarked that a_1a_2 being drawn, it must be perpendicular to the axis.

PROPOSITION XII.

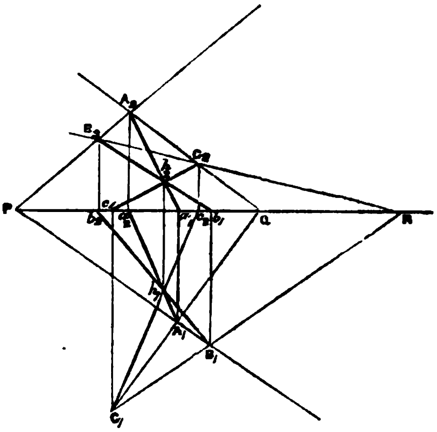
Given the traces of three planes to find the projections of their common section.

Suppose the plane P to be cut by the two planes Q and R. Find the projections of the intersections of P, Q and those of P, R: their intersections a_1, a_2 will be those of the intersection of P, Q, R.



Scholium. The following neat theorem arises from the construction for finding the intersection of three planes whose traces are given.

Let P, Q, R be three points in a straight line, and let three pairs of lines be drawn from them, forming by their intersections the triangles $A_1B_1C_1$ and $A_2B_2C_2$. Draw the perpendiculars A_1a_1, B_1b_1, C_1c_1, A_2a_2, B_2b_2, C_2c_2 to the line PQR. Join A_1a_2, B_1b_2, C_1c_2; these will intersect in the same point h_1: also join A_2a_1, B_2b_1, C_2c_1; these will intersect in the same point h_2: and finally h_1h_2 being drawn, it will be perpendicular to the line PQR.

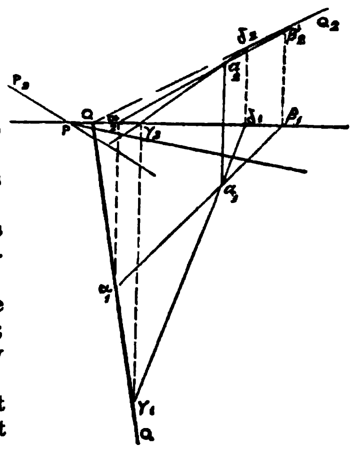


This would be extremely complex in its demonstation, by means of the ordinary plane geometry. The method of transversals will reach it.

PROPOSITION XIII.

Through a given line to draw a plane perpendicular to a given plane.

If from any point in the given line a perpendicular be drawn to the plane, and a plane be drawn through the given line and the perpendicular, it will be the one required.



Let $\alpha_1 \alpha_2$ be the point selected in the given line $\alpha_1 \beta_2$ as that from which the perpendicular shall be drawn to the plane P.

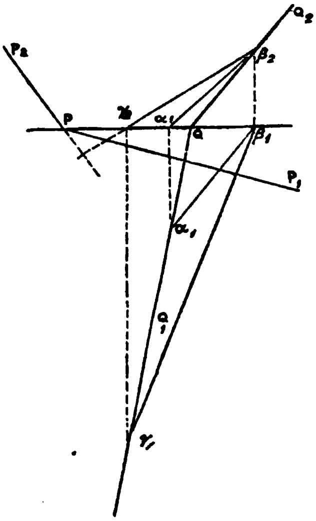
Construct (*Prop.* x.) the traces $\gamma_1 \delta_2$ of this perpendicular.

Then the plane Q whose traces pass through $\alpha_1 \gamma_1$ and $\beta_2 \delta_2$ respectively will be the one required.

For since both lines are in the plane Q, their traces are in the traces of Q; and the traces of Q are determined by the traces of those lines.

The criterion of accuracy is that $\alpha_1 \gamma_1$ and $\beta_2 \delta_2$ meet at the same point Q in the axis.

SCHOLIUM 1. The work itself becomes somewhat simplified by drawing the perpendicular from one of the traces of the given line, and finding the trace of this perpendicular on the other plane.



Thus taking β_2 in the vertical coordinate plane, drawing perpendiculars from β_2 to PP_2 and β_1 to PP_1, we get the projections $\beta_1 \gamma_1$, $\beta_2 \gamma_2$ of the perpendicular from β_2, and its other trace γ_1 as usual.

Then $\gamma_1 \alpha_1$ will be the horizontal trace; and Q, the point in which it cuts the axis, is a point in the vertical trace, which with β_2 gives that trace.

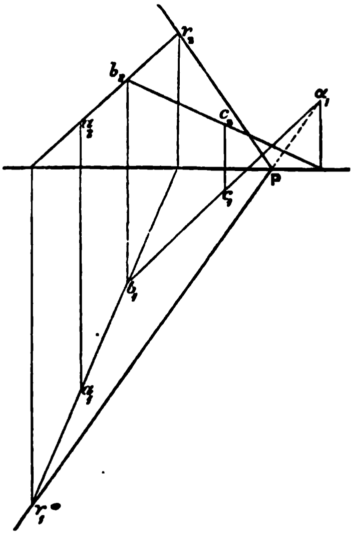
SCHOLIUM 2. If γ_1 coincide with α_1, the given line itself is perpendicular to the plane P; and the problem is indeterminate, as we know it should be, since every plane through this line is then perpendicular to P. All lines drawn to meet in the axis from these traces will then be the traces of planes perpendicular to the given plane P.

PROPOSITION XIV.

Through three given points, through two given lines which meet, or through a given line and a given point to trace a plane.

(1.) If three points be given, we can draw lines through them two and two, and the traces of all these lines will lie in the traces of the plane through them. But the traces of the plane itself will also be given, if two of the three traces of the lines be given upon one coordinate plane, and the trace of one of them upon the other.

Let $a_1\,a_2$, $b_1\,b_2$, $c_1\,c_2$ denote the three given points: find $\gamma_1\gamma_2$ the horizontal and vertical traces of *ab*, and α_1 the horizontal trace of *bc*. Draw $\alpha_1\,\gamma_1$ meeting the axis in P, and draw $P\,\gamma_2$. Then $\gamma_1\,P$, $P\,\gamma_2$ are the horizontal and vertical traces of the plane through *a*, *b*, *c*.



It is obvious that the particular traces to be selected for the determination of the plane must be regulated by the manner in which it will cause the lines of the drawing to fall, with respect to distinctness of operation and to the limits of the drawing itself.

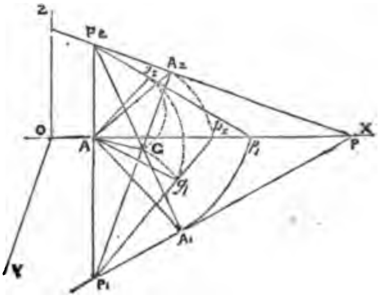
(2.) Find the traces of one line, and one trace of the other, and operate as before. If these be not convenient as to position, take three points any way, one in one line and two in the other, so that they may give a compact solution. This process is essential when the projections make very acute angles with the axis, or are parallel to it. It will generally be most advisable to cut the lines by profile planes. The construction by this method is sufficiently simple.

(3.) Take any two convenient points in the given line, and operate as in the preceding case. The traces will usually be the most convenient; but if these fall beyond the limits, take any other two.

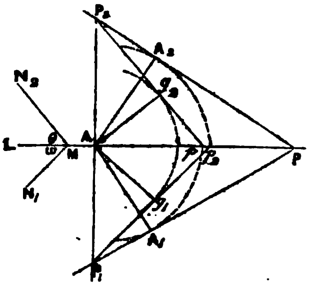
PROPOSITION XV.

A plane is given by its traces, to find its inclination to the coordinate planes.

Let the eidograph be cut by planes through any point A in the axis, respectively perpendicular to the traces PP_1, PP_2 of the given plane, cutting the plane P in P_2A_1 and P_1A_2 and the coordinate planes correspondingly. Also, make $Ap_1 = AA_1$ and $Ap_2 = AA_2$, and join $P_2\,p_1$ and $P_1\,p_2$. Then, since the plane $P_2\,A_1\,A$ is perpendicular to the trace PP_1, the lines AA_1 and $A_1\,P_2$ are perpendicular



to it (*Pls.* II. 2); and hence they measure the inclination of P to the horizontal plane. Also, since P_2AA_1 and P_2Ap_1 are right angles (for the plane P_2AA_1 is evidently at right angles to the horizontal plane), and the sides P_2A, AA_1 are equal to P_2A, Ap_1, the hypothenuse P_2p_1 is equal to P_2A_1, and the angle P_2p_1A to the angle P_2A_1A, or to the inclination of P to the horizontal coordinate plane. In precisely the same manner it appears that P_1p_2A, is its inclination to the vertical coordinate plane. Whence the construction in orthograph :—



Draw any line perpendicular to the axis, cutting the traces of P in P_1, P_2, and the axis in A; from A draw AA_1, AA_2, perpendicular to the traces; set off $Ap_2 = AA_2$ and $Ap_1 = AA_1$; and join P_2p_1 and P_1p_2: the angles P_2p_1A and P_1p_2A will be the measures of the inclinations of P to the horizontal and vertical coordinate planes respectively.

COR. 1. The perpendiculars AA_1 and AA_2 are the projections of the perpendicular from A to the plane P.

COR. 2. If, in the eidograph P_2A_1 and P_1A_2 intersect in G, AG will be the perpendicular from A to the plane P, and perpendicular to each of the lines P_2A_1 and P_1A_2. Wherefore, when the plane P_2A_2A is turned on the vertical coordinate plane, by revolving about AP_2, this perpendicular will take the position of a perpendicular Ag_2 from A upon P_2p_1. In the same, when turned upon the horizontal plane, AG will take the position of a perpendicular Ag_1 from A upon P_1p_2. These two perpendiculars Ag_2, Ag_1 are therefore equal to each other in the orthograph.

PROPOSITION XVI.

Through a given point to trace a plane which shall make given angles with the coordinate planes.

(Figures to preceding Proposition.)

Let us first suppose the given point to be in one of the coordinate planes, as P_2 in the vertical; and let θ and ϖ be the respective angles which the required plane is to make with the vertical and horizontal coordinate planes.

Draw P_2A perpendicular to the axis of indefinite length; and draw P_2p_1 parallel to N_2M: about centre A describe the circle p_1A_1: draw the perpendicular Ag_2 upon P_2p_1, and about centre A describe the circle g_1g_2: draw a tangent to this circle parallel to MN_1, meeting P_2A in P_1: draw a tangent P_1P to the circle A_1p_1: and, lastly, join PP_2. Then the plane P_1PP_2 makes the given angles with the coordinate planes.

When the given point is not in a coordinate plane, assume some point in one of them, construct as just directed, and through the given point draw a plane (*Prop.* VI.) parallel to that thus constructed.

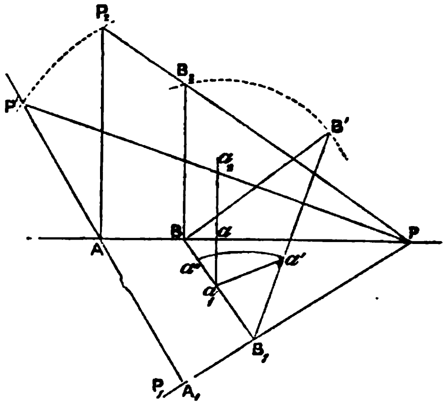
The truth of this construction will be evident from comparison with the preceding, taken in connexion with the second corollary.

SCHOLIUM. As a visual verification of the work itself, we may describe the circle p_1A_1 about centre A: which ought to touch the line PP_2.

PROPOSITION XVII.

Given the traces of a plane, and the projections of a point in it, to turn the plane itself, and the point in it upon either coordinate plane, and to find the angle made by the traces in eidograph.

We shall turn the system on the horizontal plane for exemplification; the operation on the vertical being precisely similar.



Let P_2PP_1 be the plane. Cut the system by any plane perpendicular to the horizontal trace in AA_1, and AP_2. Produce AA_1 and about P as centre with radius PP_2, describe an arc to meet AA_1 produced in P′. Then P′P being drawn, it will be the trace PP_2 in eidograph, after revolution on the horizontal plane.

For P_2A_1P and $P'A_1P$, have the angles P_2A_1P, $P'A_1P$ right angles; they have also one side A_1P common, and the hypothenuses equal. Whence P_2PP_1 is equal to $P'PA_1$ in orthograph, as here constructed.

Again, for the revolution of any point a_1a_2 in the plane P upon the horizontal coordinate plane. [Though for expression's sake we have noted the two projections a_1 and a_2, only one of them is really given in any properly proposed problem, since the condition of the given point being in the given plane is adequate to the exhibition of the second projection a_2 or a_1, according as a_1 or a_2 is given actually.]

Through a_1 (the given projection of the given point) draw BB_1 perpendicular to PP_1, and BB_2 perpendicular to the axis meeting the vertical trace of the given plane in B_2: draw BB' and a_1a' parallel to PP_1: make $BB' = BB_2$, and join B_1B', cutting a_1a' in a': and finally make $B_1a'' = B_1a'$. Then a'' is the given point after revolution on the horizontal plane.

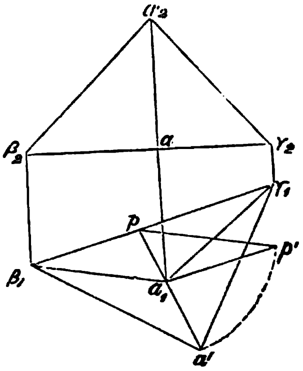
For it is clear that B_1BB' is the triangle B_1BB_2 of the eidograph after revolution about B_1B upon the horizontal plane; and hence that B_1a' is the real distance of the given point from the point B_1. Also, that if instead of turning about BB_1, we turn about PP_1, the point a' will not change its distance from B_1: that is, it will be found at a''.

Scholium. The method just given, if taken in the abstract, is the simplest mode of proceeding; but in some cases the data may be so related as to render some other process more convenient. The revolution of the plane itself may be effected by finding the distance between any two points, one in each trace, whether they be in a plane perpendicular to one of the traces or not: and then upon the axis of revolution constructing a triangle whose sides shall be that distance so found, and the intercepted portion of the other trace. See Case 2 of the next proposition for illustration.

PROPOSITION XVIII.

To construct the eidographic angle of inclination of two given straight lines.

The general principle that runs through every construction of this problem is—to exhibit the two lines after revolution on one of the coordinate planes, the axis of revolution being one of the traces of the plane which contains them.

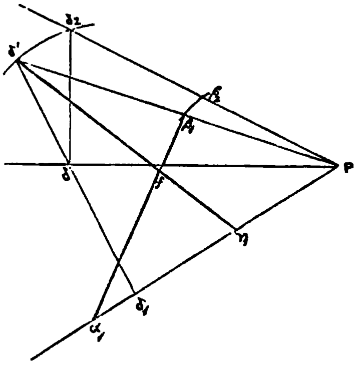


(1.) Let the lines be given by means of their horizontal traces β_1, γ_1, and the projections of their point of intersection a_1, a_2.

Join $\beta_1 \gamma_1$, and draw $a_1 p$ perpendicular to it and $a_1 p'$ parallel. Make $a_1 p'$ equal to aa_2, and describe a circle about p with pp' as a radius to cut $a_1 p$ in a'. Then, drawing $\beta_1 a'$, $\gamma_1 a'$, the angle $\beta_1 a' \gamma_1$ is the required inclination of the given lines.

(2.) Let the lines be given by means of their traces on both coordinate planes.

In this case let $\alpha_1 \beta_2$, $\gamma_1 \delta_2$ be the lines exhibited by their traces. Through the traces draw the traces of the plane which contains them, meeting the axis in P. Draw $\delta_2 \delta$ perpendicular to the axis, and $\delta \delta_1$ perpendicular to the trace α_1 P, which produce. From P as centre, describe arcs $\delta_2 \delta'$ and $\beta_2 \beta_1$, the former meeting $\delta\delta_1$ in δ', and the latter cutting Pδ' in β_1. Then δ' and β_1 are the traces β_2 and δ_2 on the horizontal coordinate plane after revolution. Whence the lines $\delta' \gamma_1$ and $\beta_1 \alpha_1$ being drawn, they will be the given lines, and the angle at *f*, their intersection, is that which we were required to construct.



In reference to both the preceding cases, we may remind the student that he may draw *any two lines* parallel to the given ones, through the same point, so as to bring the work into the desired position. The angle contained by these is equal to the angle of the two original lines (*Pls.* I. 7).

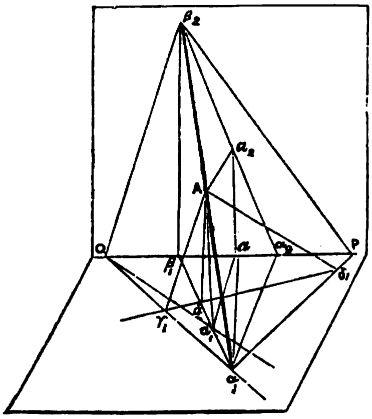
(3.) If the given lines be not considerably remote from parallelism to the axis, the traces of them will fall beyond the limits of the drawing. In this case we must proceed somewhat differently. The following is the simplest process that has been devised for the purpose.

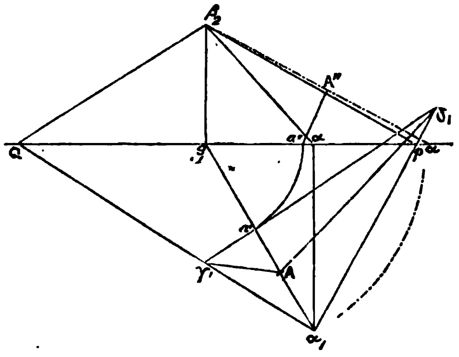
Cut them by profile planes (as in *Case* 3, *Prop.* VII.), and find the traces of the plane which contains them. Turn this plane upon one of the coordinate planes, and likewise two points of each of the given lines (those used in the profile will be most convenient in practice): then these lines give the inclination sought, as also will any parallels to them, which will sometimes suggest an advantageous construction.

PROPOSITION XIX.

Given the orthograph traces of two planes, to construct their eidographic inclination.

Let the given planes, P, Q in the eidograph intersect in the line $\alpha_1\ \beta_2$,





and let a plane be drawn perpendicular to it through any point A, cutting the horizontal coordinate plane in $\gamma_1\ \delta_1$; and the given planes in $A\gamma_1\ A\delta_1$, and the projecting plane $\alpha_1\ \beta_1\ \beta_2$ in Aa'. Then it is required in the problem to construct an angle equal to $\gamma_1\ A\delta_1$.

Now since the plane $\gamma_1\ A\delta_1$ is perpendicular to the line $\alpha_1\ \beta_2$, this line is perpendicular to $A\gamma_1$, Aa_1, and $A\delta_1$ (*Pls.* II. 2).

Again, since $\alpha_1\ \beta_2$ is perpendicular to the plane $\gamma_1\ A\delta_1$, the plane $\alpha_1\ \beta_1\ \beta_2$ through it is perpendicular to $\gamma_1\ A\delta_1$ (*Pls.* II. 16); and since it is the projecting plane upon the horizontal plane of $\alpha_1\ \beta_2$, it is perpendicular also to the horizontal coordinate plane. Whence the plane $\alpha_1\ \beta_1\ \beta_2$ is perpendicular to $\gamma_1\ \delta_1$ the intersection of these planes; and $a'A$ in it is perpendicular also to $\gamma_1\delta_1$.

It follows from this, that if the plane $\gamma_1\ A\delta_1$ be turned upon the horizontal plane about $\gamma_1\ \delta_1$, the point A will be situated somewhere in the line $a_1\ \alpha_1$; and it only remains to find the eidographic length of $a'A$ by a plane construction. Two general cases will arise here.

(1.) When the traces of the line $\alpha_1\ \beta_2$ can be employed, that is, when they both fall within the limits of the picture.

Draw any line $\gamma_1\ \delta_1$ to cut the horizontal traces of the given planes in

$\gamma_1\delta_1$, and the horizontal projection $\alpha_1\beta_1$ of the line, perpendicularly at a'. Make $\beta_1\alpha = \beta_1\alpha_1$, and $\beta_1 a'' = \beta_1 a'$; join $\beta_2\alpha$ and draw the perpendicular $a''A''$: then make $a'A' = a''A''$, and join $\gamma_1 A'$, $\delta_1 A'$. The angle $\gamma_1 A'\delta_1$ is that required.

For, evidently, the triangle $\beta_2\beta_1\alpha$ is the eidographic triangle $\beta_2\beta_1\alpha_1$ revolved on the vertical; and $a''A''$ the perpendicular from a' upon the line $\beta_2\alpha_1$. It thus fulfils the condition already deduced.

In certain cases this construction may be a little simplified, viz.:—

(*a*.) When the figure will admit of it, the point A may be taken at β_2, in which case the perpendicular from β_1 upon $\beta_2\alpha$ takes the place of the perpendicular $A''a''$ of this construction.

(*b*.) A plane parallel to one of the given planes may be used instead of that plane itself; or, even if the work would fall more conveniently, planes parallel to both of them may be used instead of them.

(*c*.) The case when the two given planes intersect the axis in the same point renders a plane parallel to one of indispensable use, in order to avoid a long and complex construction dependent on a separate investigation.

(2.) Let the traces of the lines of section be situated so as to be incapable of being employed in the construction.

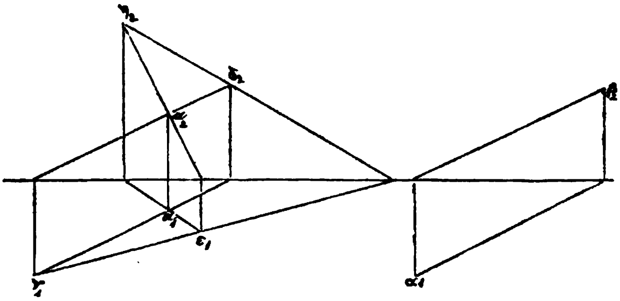
Find the projections of two points in the line of section (*Prop.* VII., *Case* 3), and draw its horizontal and vertical projections; also let a_1, a_2 be the horizontal projections of one of these points. Through this point draw a plane perpendicular to the line thus found (*Prop.* XVI.), and let it cut the horizontal traces of the planes, as before, and find the vertical projection of the point a' in which it cuts the horizontal projection of the line of section.

Lastly. Find the distance of the two points a_1a_2 and $a'a''$: this is the perpendicular from a' upon $\alpha_1\beta_2$, and the angle is found as before.

PROPOSITION XX.

Through a given line to trace a plane parallel to another given line.

Through the given line $\varepsilon_1\eta_2$ to draw a plane parallel to the given line $\alpha_1\beta_2$.

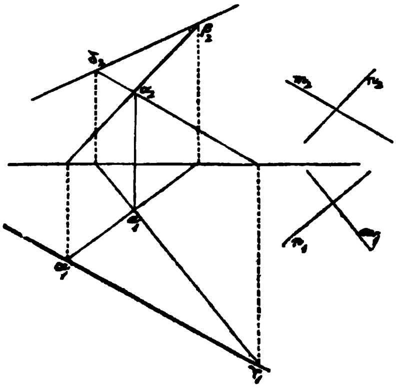


Take any point $a_1 a_2$ in $\varepsilon_1\eta_2$, and draw a line parallel to $\alpha_1\beta_2$, viz., $\gamma_1\delta_2$.

Then $\gamma_1\varepsilon_1$ and $\delta_2\eta_2$ will give the traces of the plane sought.

SCHOLIUM. In nearly the same way we may trace a plane through a given point parallel to two given lines which are not in one plane.

Let m n be the lines, and a_1 a_2 the point. Through a_1 a_2 draw the



lines parallel to the given ones respectively. Let α_1 β_2 be the traces of the line parallel to n_1 n_2, and γ_1 δ_2 those of the line parallel to m_1 m_2. Then the traces of the plane through a_1 a_2 and which is parallel to m_1 m_2 and n_1 n_2 are α_1 γ_1, β_2 δ_2.

PROPOSITION XXI.

To construct the eidographic inclination of a given line to a given plane.

1. Through any point in the given line draw a plane perpendicular to the given plane (*Prop.* XVI.).

2. Find the intersection of this plane with the given plane (*Prop.* VII.).

3. Find the angle made by this line of intersection and the given line (*Prop.* XVIII.).

This is the angle required.

PROPOSITION XXII.

Through a given point to draw a line perpendicular to a given line.

1. Find the traces of the plane through the given point and given line (*Prop.* XIV.).

2. Find the traces of a plane perpendicular to the given line which shall also contain the given point (*Prop.* X., *Scholium* 2.).

3. The intersection of these two planes being formed, it is evidently the perpendicular required.

PROPOSITION XXIII.

To draw a line which shall be perpendicular to two given lines which are not in the same plane.

1. Through each of the given lines draw a plane parallel to the other (*Prop.* XX). These planes will be parallel.

2. Through each line draw a plane perpendicular to that which has been made to pass through the other line (*Prop.* XXII.).

3. Find the intersection of these perpendicular planes (*Prop.* VII.).

This line will be perpendicular to both the given lines.

EXERCISES.

1. Construct the traces of a plane which passes through the three points (−1, 2, 3), (1, 3, 5), (2, 0, 3).

2. Given the horizontal projection of an equilateral triangle, and the vertical projection of one of its sides to construct the vertical projection of the entire triangle.

3. Given the horizontal and vertical projections of a tetrahedron to assign the conditions that must be fulfilled amongst the parts of the projection to render it a regular tetrahedron.

4. Through a given point to draw a plane parallel to two given lines. In what case does this problem become indeterminate?

5. Find a line equal to the distance between two given parallel planes.

6. Given the projections of a point and a line, to construct those of a line drawn through the given point perpendicular to the given line.

7. Through a given point to draw a line parallel to two given planes.

8. Draw a plane which shall make equal dihedral angles with two given planes:—

(1.) The plane to be drawn through the intersection of the given ones;

(2.) Through any given point.

9. Given the projections of a finite line to divide it in any given ratio.

10. Find the angle contained between the traces of a given plane.

11. Draw a plane through a given line which shall make a given angle with either plane of projection; and another which shall make equal angles with both planes.

12. Given a trihedral angle and a point; to show the supplementary trihedral angle whose vertex is in the point.

13. Given the projections of any one radius of a circle to find the circle after revolution on the vertical plane.

14. Given the projections of three points in the circumference of a circle to construct its radius.

15. Through a given point to draw a line which shall meet two given lines.

16. Through a given point in a given plane to draw that line in the given plane which shall have the greatest inclination to the horizon, and that which shall have the least inclination to the vertical plane.

17. A building is to be erected on a declivity, but its foundation is to be horizontal, having the plan of the foundation, to trace on the surface of the ground the outline of the cutting.

18. A triangle on a plane is given: to find the position of a plane upon which if it be projected orthographically it shall become similar to a given triangle.

SECTION IV.

THE CONE, CYLINDER, AND SPHERE.

These are the only surfaces which are usually brought into engineering practice, except on very special occasions. When such occur, a more extended series of studies of the Descriptive Geometry must be entered upon, and the engineer must have recourse to treatises expressly and solely devoted to the subject.

The problems here given will amply display the general character of this class of researches; and it will be seen that they differ but little, as to general method, from those already developed at length.

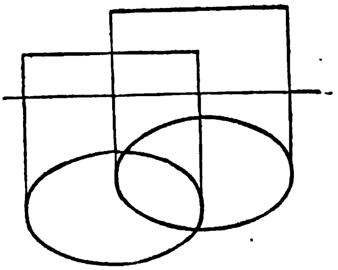
It will be desirable, however, to prefix a few notes on the mode of designating *surfaces* to suit this method of construction.

(1.) As has been already shown, a plane is designated usually (because most conveniently) by its traces on two coordinate planes. Indeed one trace and a point on the other is sufficient; and when the plane is given by means of other specific conditions, it is still the best way of proceeding to find the traces of that plane as the preliminary step.

(2.) The cylinder being the extreme case of the cone (where the vertex is infinitely distant and the edges of the cone consequently parallel), the two surfaces will be designated by nearly similar methods, and the constructions will partake in a great degree of the same character.

(*a*) The cone and cylinder may each be defined (that is, their elements represented) by means of the projections of two sections parallel to either of the coordinate planes. Thus:—

When these two sections are equal, the figure is a cylinder; and when unequal, a cone. They are necessarily similar whether equal or not (*Prop.* XX. *Chap.* I. *Pls. and Sols.*).



(*b*) The most usual way of defining the cone and cylinder is by means of the horizontal trace of the surface and the projections of the axis. The horizontal trace (the vertical might have been taken instead) of the cone or cylinder is most frequently taken as a circle, in which case the centre of the circle is a point in the axis; and only one point more is required for fixing the axis. In the cone, that one point is invariably the centre of the surface or vertex; in the cylinder, any point whatever.

Even when the cone or cylinder is given by the first method, it is most convenient, and sometimes necessary, to reduce the conditions to the second method of definition.

(3.) The sphere is defined by means of its radius given in magnitude, and the projections of its centre given in position.

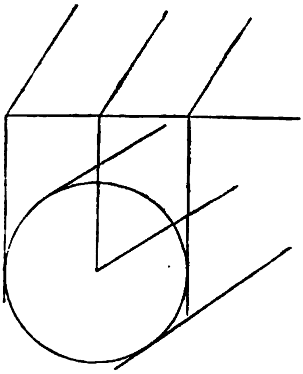
Sometimes it may happen that when the sphere, instead of being directly given, results from prescribed constructions, it may present itself by means of some other elements. Those elements can however always be, and indeed always are, reduced to the condition stated above, anterior to any further constructions which involve it.

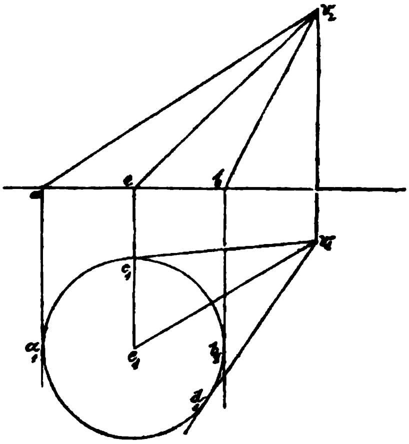
Of the ellipsoid, which is the surface next to these as regards elementary character, nothing can be given in this course, and hence the mode of defining it may be properly omitted.

PROPOSITION I.

Given one trace and the projections of the axis of a cone to find the extreme limits of its projections on both planes.

Let v_1 v_2 be the vertex, e_1 the centre of the circular trace, and the circle as in the figure.





Draw the tangents a_1 a, $b_1 b$ perpendicular to the axis; and join av_2, bv_2. The space $av_2 b$ comprehends the projections on the vertical plane of all the points of the cone between the vertex and trace inclusive. If points beyond these limits be taken into consideration, they will still lie between the prolongations of these lines taken suitably to the case.

Draw the tangents v_1 c_1, v_1 d_1 to the trace. Then the horizontal projections of all the points in the cone between the vertex and the base inclusive will lie in the space bounded by the circular arc $c_1 a_1$ d_1 and the tangents $c_1 v_1$, d_1 v_1. The extensions of the tangents $c_1 v_1$, $d_1 v_1$ include the projections of all points in the extended cone, as in the former case.

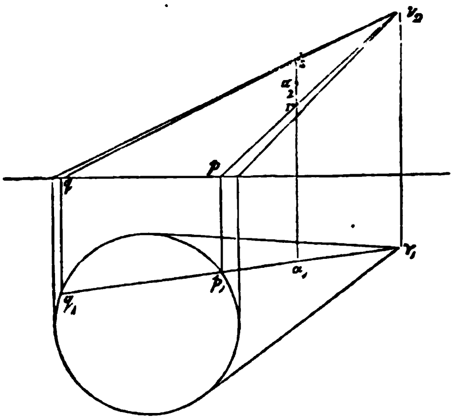
In the cylinder the process is precisely similar, except that the limiting lines are parallel instead of converging to a point, as in the unlettered figure.

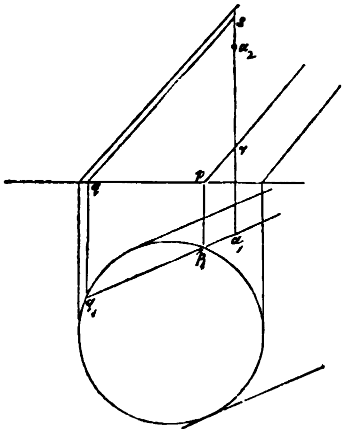
Scholium. The figures contain also the construction for the projections of the axis, which the student will explain.

PROPOSITION II.

Given the elements of a cone or cylinder and the projections of a point, to find whether that point be on the surface or not.

(1.) If the projections a_1, a_2 of the point do not both lie within the regions prescribed by the preceding problem, the point cannot be on the surface.





(2.) If they be situated within the limits, each will be the projection of some two points of the surface (since a line may cut the surface in two points); the horizontal projection being that of one pair of such points, and the vertical that of another pair of such points.

Suppose, then, that a_1 is the horizontal projection of a point on the surface: we are required to find whether a_2 is the vertical projection of the same point.

Draw $v_1 a_1$ ($v_1 v_2$ being the vertex) to meet the trace in p_1, q_1. This is the horizontal projection of the edge of the cone through that point; and p_1, q_1 are the traces of the two edges, either of which might contain the point in question. Find, as usual, their vertical projections pv_2, qv_2, and let them cut $a_1 a_2$ in r and s. Then, if either r or s coincide with a_2, a_2 will be the vertical projection of the point on the surface whose horizontal projection is a_1: if not, the point $a_1 a_2$ is not upon the given surface.

SCHOLIUM. If a_2 fall between r and s, the point $a_1 a_2$ is within the cone or cylinder; if without, without.

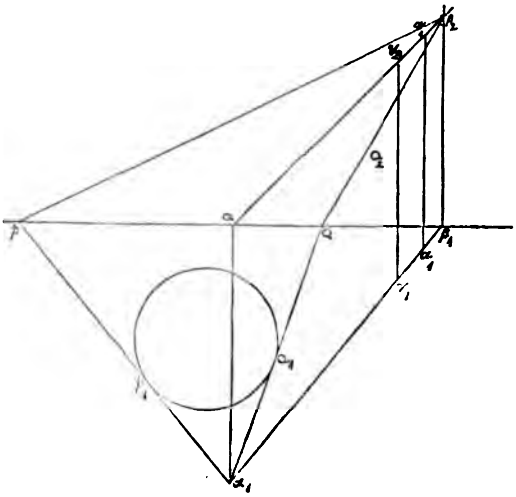
In this process we have solved the problem: *given one projection of a point situated on a given cone or cylinder to find the other.*

There is however another case left for the student: when the vertical projection a_2 is given, to find a_1.

PROPOSITION III.

Through a given point to draw a tangent plane to a given cone or cylinder.

(1.) Let $v_1 v_2$ be the given vertex of the cone, and $a_1 a_2$ the given point. Both these are in the tangent plane, and the tangent plane contains both the traces of the line through them. Let the horizontal trace α and the vertical β_2 be found as usual.



Again, since the plane touches the cone, its horizontal trace is a tangent to the horizontal trace of the cone. Whence drawing α_1 P, α_1 Q to touch the horizontal trace of the cone in P_1, Q_1, these will be the horizontal traces of the two planes through $a_1 a_2$ touching the cone.

Lastly, as the plane also passes through β_2, we have only to draw $P\beta_2$, $Q\beta_2$, which will be the vertical traces required.

(2). When the point $a_1 a_2$ is in the surface of the cone, the two tangent planes coalesce. The process, though slightly modified, is very nearly the same; the difference merely arising from α_1 falling in the trace of the cone. The tangent to the trace of the cone at that point is the horizontal trace of the tangent plane; and the vertical trace is found as before.

(3.) For the cylinder, the only difference in the construction is, that instead of drawing the line $a_1 a_2$, $v_1 v_2$ to a given point, we must draw it parallel to the given axis, or one of the given generatrices.

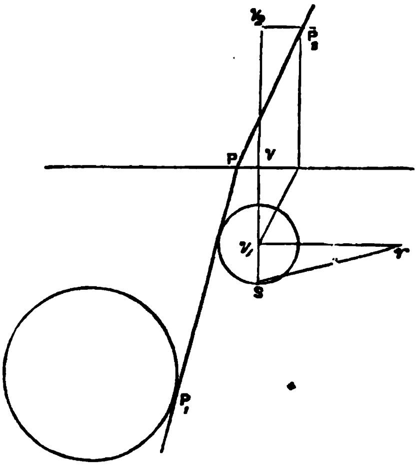
PROPOSITION IV.

To draw a tangent plane to a given cone, which shall make a given angle with the horizontal plane of projection.

Using the former notation for the data, draw $v_1 r$ parallel to the axis, and make $v_1 r = vv_2$, and the angle $v_1 rs$ the complement of the given inclination; with centre v_1 and radius $v_1 s$ describe a circle; and to this and the trace draw a common tangent $P_1 P$. This will be the horizontal trace required, but the proof is left as an exercise for the student.

To find the vertical trace, we have only to recollect that the plane (P) passes through v v_2, and that we have found the horizontal trace

$P_1 P$. The construction is therefore reduced to a former problem; and is, moreover, suggested by the lines drawn in the figure.

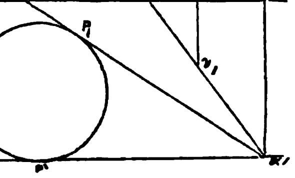


Scholium. When the circle about v_1 falls wholly without the trace of the cone, there will be *four* solutions; when it touches externally, there will be three; when it cuts, there will be two; when it touches internally, there will be one; and when it falls wholly within, there will be none.

The construction for the cylinder will only differ from the preceding in minor particulars; which, however, the student is required to point out.

PROPOSITION V.

A cone and a line through its vertex are given, to draw through the line a tangent plane to the cone.



All planes through the given line (whether they are tangent to the cone or not) pass through the traces of that line. Let then $v_1\ v_2$ be the vertex of the cone, and α_1, β_2 the traces of the given line through it.

Draw $\alpha_1\ P_1\ P$ to touch the trace of the cone; this will be one of the traces of the plane. The other trace passing through β_2, the line $P\beta_2$ is that trace itself. The plane is hence determined.

Scholium. If the line $\alpha_1\ \beta_2$ fall *within* the conical surface, no such tangent plane can be so drawn; if it fall *on* the surface, one only can be so drawn; and if wholly *without*, there can be two.

There is no corresponding case in respect to the cylinder.

PROPOSITION VI.

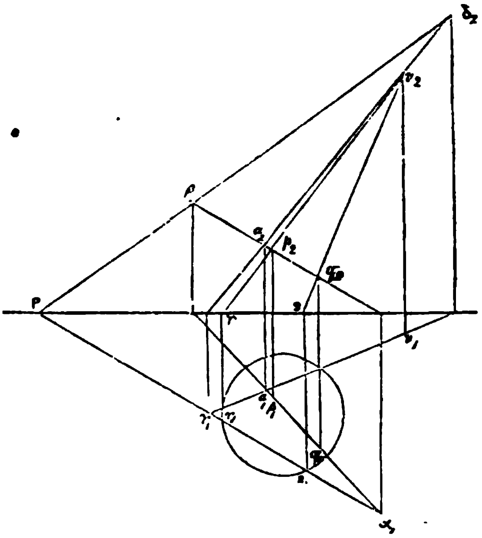
To find the intersections of a given plane with a given cone or cylinder.

The sections of the cone may be either ellipses, parabolas, or hyperbolas, including all their varieties (*Conic Sections*, p. 221). Those of the cylinder only the ellipse and its varieties, and straight lines. Generally, for the determination of these, five elements are required. Thus far with respect to the sections themselves; but in the present case it is moreover required to find their orthographic projections on the coordinate planes, which, however, does not alter the *species* of the figures, the ellipse still being projected into an ellipse, a parabola into a parabola, and an hyperbola into an hyperbola.

The most obvious mode of proceeding is—to take any five convenient generatrices of the cone or cylinder, and find the projections of their intersections with the given plane. The conic sections traced through the five points thus obtained on the coordinate planes will be the projections required. The figure, however, which contains the entire process is too much crowded with lines to be given here; but it is easily drawn, with proper care, upon a larger scale, and the whole process is too intelligible to need further remark.

PROPOSITION VII.

To find the intersection of a given line with a given cone or cylinder.



Find α_1, β_2 the traces of the given line; take any point a_1 a_2 in it, and find the traces γ_1, δ_2 of the line through a_1 a_2 and v_1 v_2. The plane whose traces are α_1 γ_1 and β_2 δ_2 passes through the given line and the vertex of the cone, and cuts the cone in those two of its edges which pass through the intersections of the given line and given cone.

Let r_1, s_1, be the intersections of the traces of this plane and the cone. Find as usual (and as indicated by the lines in the figure) the vertical projections rv_2, sv_2 of these edges, and let them meet the vertical projection of the given line in p_2, q_2. Draw $p_2\,p_1$, $q_2\,q_1$ perpendicular to the axis, meeting the horizontal projection of the given line in p_1, q_1. Then $p_1\,p_2$ and $q_1\,q_2$ are the points in which the line meets the cone.

In the case of the cylinder, the only difference is—that in finding the line through the assumed point $a_1\,a_2$ in the given line, drawn to a given point, $v_1\,v_2$, that line is to be drawn parallel to the given axis (or a generatrix) of the given cylinder. The work is *slightly* lessened, but the principle is the same.

PROPOSITION VIII.

Through a given line to draw a tangent plane to a given sphere.

On account of the multiplicity of the lines which enter into all the steps of the construction, the process can only be described here; but the student will be required to perform them in detail.

(1.) Through the centre of the given sphere draw a plane perpendicular to the given line, finding the projections of the point of intersection.

(2.) Find the distance of this point from the centre.

(3.) Construct (separately) a right-angled triangle having the distance in (2) for its hypothenuse and the radius of the given sphere for one of its sides, so as to find the angle opposite to this radius.

(4.) Through the given line and centre of the sphere describe a plane.

(5.) Through the given line describe a plane to make with that in (4) an angle whose profile is equal to that determined in (3).

This will be the plane required; or, rather, there will be two, one on each side of the plane (4).

PROPOSITION IX.

To find the intersection of a given sphere with a given plane.

Generally the projections will be ellipses; one projection being a circle only when the given plane is parallel to one of the coordinate planes, and the other a limited straight line. This case is so simple that it may be left without further instruction for the student's exercise.

(1.) When the given plane is oblique to both coordinate planes, find its traces.

(2.) From the centre of the sphere draw a line perpendicular to the given plane. The projections of the point of intersection of the line and plane will be those of the centres of the ellipses into which the sectional circle is projected.

(3.) The major axis of each ellipse is parallel to the trace of the given plane on the plane of projection, and equal to the radius of the sectional circle.

(4.) The minor axis perpendicular to this is the same radius reduced in a given ratio. (*See* Conic Sections, p. 221.)

(5.) The radius of the sectional circle is thus found:—

> Find the length of the perpendicular from the centre of the given sphere on the given plane: form a right-angled triangle having the given radius of the sphere for its hypothenuse and the perpendicular for one of its sides; the other side will be the sectional radius.

SCHOLIUM. The problem of finding the intersection of two given spheres is at once reducible to this, since that section is a circle whose plane is readily determinable.

PROPOSITION X.

To find the intersections of a given straight line with a given sphere.

(1.) Draw a perpendicular from the given centre to the given line; and assign the point of intersection, and the distance of the intersection from the centre.

(2.) Construct a right-angled triangle (separately) whose hypothenuse is the radius of the given sphere and one of its sides the distance just found. The other side is the semi-length of the segment of the intercepted line.

(3.) Describe on this line as hypothenuse another right-angled triangle having one of its angles equal to the inclination of the given line to either plane of projection. The side adjacent to this angle is the semi-length of the projection of the intercepted line on that plane.

(4.) The remaining part of the construction (one single step) is obvious.

SECTION V.

SHADOWS.

This is one of the simplest applications of Descriptive Geometry, at least as far as the subject is of any use in engineering and architectural drawings. If, however, pursued into all its conceivable applications, it becomes extremely complex. Only the simplest and most elementary problems have been deemed necessary here, but a full comprehension of these will be fully adequate to enable the draughtsman to project any shadow that can occur in his practice.

A few general but brief remarks may be usefully made at the outset.

1. Although the method of determining the shadows of objects here given is *perfectly general*, and adapted to all circumstances of the relative position of the object, the light, and the surface upon which the shadow is thrown; yet in fixing upon them in the case of sun-light, it is the universal English practice to select one specific position of the sun with respect to the principal face of the object, that face being also parallel to the vertical plane of projection. I know, indeed, of no merely professional work published in England in which the general method is given, and only one, and that only semi-professional, in our

language:* though such treatises, illustrated with ample examples, are frequent on the Continent. It is, however, always preferable to understand the general principles that run through an entire system, rather than to be confined to the special cases of them which the study of mere convenience may render it advisable to adopt in practice.

2. This special rule to which reference has been made is that wherein the luminous rays are said "to make angles of 45°"—a phrase, certainly, which in itself does not contain a very definite meaning. It is, however, this:—

The direction of the sun's ray is such that if the line be projected in the horizontal and vertical planes, its projections will make, with the axis, half a right angle; the right angle, Trig., Vol. I., p. 213, being designated as 90°, and hence 45° denotes half a right angle.

3. There are, indeed, good reasons offered for this choice as a general rule, especially where the simplicity of the work and the *effect* of the drawings are any way objects of consideration. The simplification will be best seen by a few trials, which make a stronger effect on the mind than mere verbal description. They belong, too, rather to the province of the practical instructor than to the development of fundamental principles.

4. The *effect* is two-fold. It gives a breadth of shadow agreeable to the eye, midway between the "meagre" and the "heavy." It also tells to the professional eye the prominence of the object which casts the shadow, without reference to the projection on the other plane, since the shadow's breadth is equal to that prominence. A shadowed elevation is, therefore, tantamount to an unshadowed plane and elevation together.

5. Although direct constructions admitting the feasibility as well as possibility of constructing the conic sections from a suitable number of elements are theoretically and practically elegant, yet in actual drawings they are seldom resorted to, except in the simplest cases. For the shadows on pillars, niches, as well as for shadows cast by them on planes, the best method in practice is to find the separate shadows of points at convenient distances, and then trace a curve through each of these. In this case it is always advisable to draw planes through the points of the original, parallel to the ray and perpendicular to the horizontal plane of projection.

6. Other suggestions might be added were they not in some degree foreign to a mere sketch like the present.

7. Of the degrees of intensity of the shadow in different regions within its limits, nothing can be said that would be of the least practical use. The problem, indeed, is a mixed one—partly physical and partly geometrical; and though capable of a perfect mathematical solution, we are yet left without any means of constructing the solution; for we have no means practically of *making* a shadow of any specified intensity. We trust to the eye after all. Even could we by mechanical means "lay on" a shadow of given intensity at a given point, our modes of shading are still incompatible with any approach

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* "A Treatise on Shades and Shadows, and Linear Perspective." By Charles Davies, Professor of Mathematics in the United States Military Academy. New York, second edition, royal 8vo. 1838. This work deserves my most cordial recommendation to the English reader who wishes to enter fully into the subjects treated in it.

towards perfect gradation of intensity. We cannot shade by a point at a time, or even by a line at a time.

DEFINITIONS.

1. The portion of any figure which is exposed to the *direct* light of any luminary is called the *illuminated part*, or the *light part* of the figure.

2. The portion of the figure which does not receive *direct* light from the luminary is called the *shaded* part of the figure, or the *shade* of the figure.

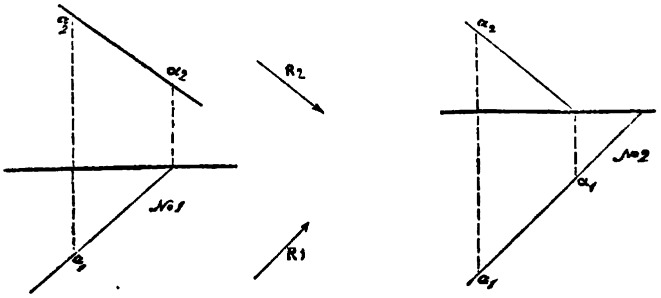
3. The portion of any other surface from which the direct light is intercepted by a given figure is called the *shadow* of that figure.

4. The boundary of the light and shade upon any figure is called the *line of shade*.

5. When the light, instead of emanating from a single point, comes from all the points of a luminous body to which the figure is exposed, and produces a variety of shadows of different intensities, the band between the absolute shadow and absolute enlightenment is called the *penumbra*.

PROPOSITION I.

To find the shadow of a point, the rays of light being parallel to a given line.



Let a_1 a_2 be the given point, and R_1 R_2 the line to which the light is parallel; then the line through a parallel to R will pierce one, and only one, of the coordinate planes in the visible or first region of space. We have, therefore, only to construct the *visible* trace of that line.

This is the same problem as that in Descriptive Geometry: *to draw through a given point a, a line parallel to a given line* R.

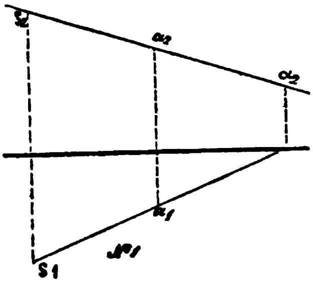
In *Fig.* 1, the shadow falls on the vertical, and in *Fig.* 2 on the horizontal coordinate plane.

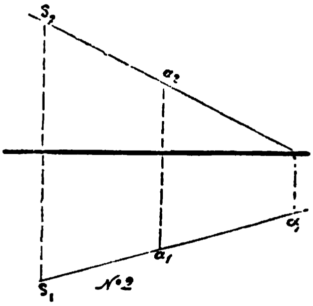
SCHOLIUM. This is the foundation of the entire system of *shadowing* in architectural and mechanical drawings.

PROPOSITION II.

To find the shadow of a point, the light emanating from a given point.

Let *s* be the luminous origin, and *a* the given point: then the problem is identical with that in Descriptive Geometry, which requires us to find the visible trace of a line passing through the two given points *s* and *a*.





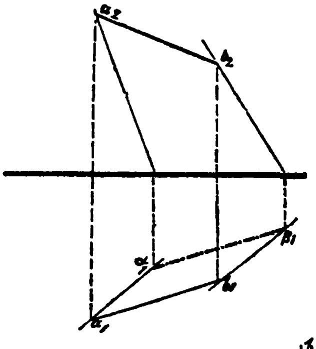
In Fig. 1, the shadow falls on the vertical, and in Fig. 2, on the horizontal coordinate plane.

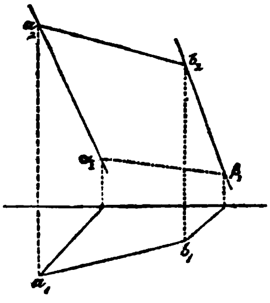
Scholium. This is the foundation of shadowing from artificial lights, as in interior night representations.

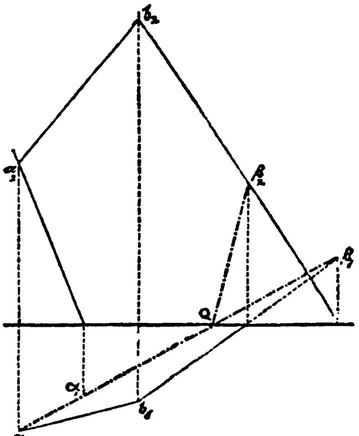
PROPOSITION III.

To find the shadow of a given line, the rays being parallel.

1. Let *a b* be the given line, terminated at *a* and *b:* then the







shadows of the extremities being joined (these being found by *Prop.* I.), we shall have the shadow required: the shadow falls *wholly* upon and within the horizontal coordinate, as in Fig. 1.; or *wholly* on the vertical, as in Fig. 2.

2. Should, however, the trace of the shadow line through one extremity of the line fall upon one plane and the other on the other, a different method, or rather an extended method, must be employed.

Since the ray through the given line at either extremity and the line itself are in one plane, the trace of that plane upon the coordinate planes will be the *position* of the shadow; and its limits have already been found. It is hence only necessary to find in addition to α_1 and β_2 the horizontal trace β_1 of the ray through *b;* which enables us to find the intersection Q of the shadow plane with the axis. The shadow will hence be composed of the two parts α_1 Q, Q β_2.

Scholium 1. Should the position of the given line and the ray render it more convenient, one of the traces of the given line *ab* will answer the purpose of finding Q; for both its traces are also in the traces of the shadow plane.

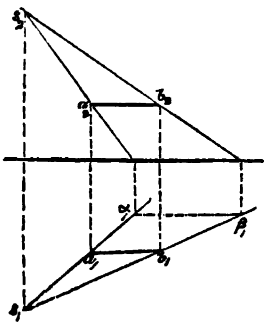
Scholium 2. If the shadow of the line *ab*, merely *given in position*, without reference to its extremities be required, the process will be somewhat simplified in practice by finding the traces of *ab*, and of a line parallel to the ray through one of the traces of *ab*.

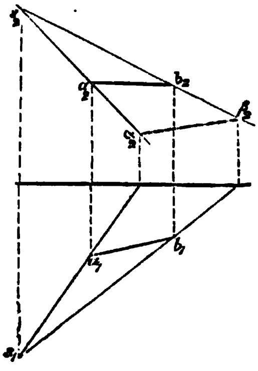
PROPOSITION IV.

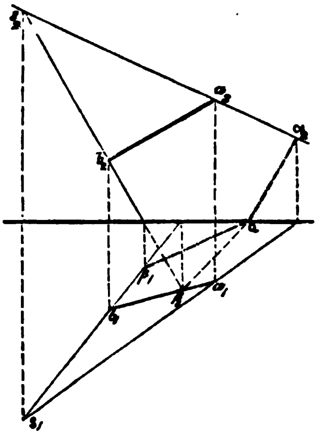
To find the shadow of a given line, the rays emanating from a given point.

In this case let *ab* be the given line, and *s* the given origin of light: then the extremities *a*, *b* are to be considered as points through which lines are drawn from *s* to meet the coordinate planes: that is, the visible traces of *sa*, *sb* are the shadows of *a* and *b*.

In Fig. 1, these are both on the horizontal coordinate plane, and the shadow is $\alpha_1 \beta_1$; in Fig. 2, they are both on the vertical, and the shadow is $\alpha_2 \beta_2$; whilst in Fig. 3, one visible trace is on each plane, and, as in the preceding proposition, another trace of one of the shadow lines, as β_1 is necessary for finding Q the point in which the shadow cuts the axis.

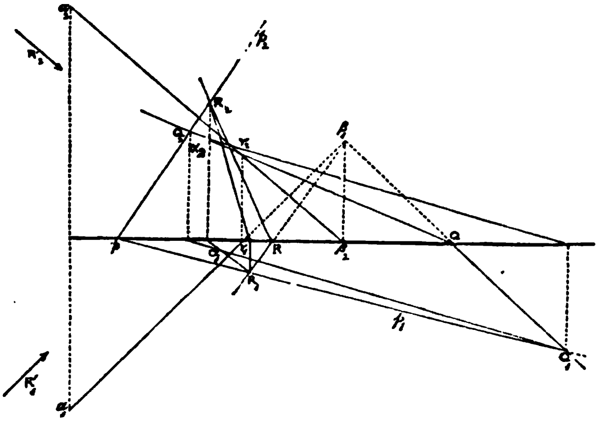






PROPOSITION V.

To find the shadow of a given point in a given plane, the rays being either parallel or emanating from a given point.



1. Let a be the given point, P the given plane, and R′ the ray: find the traces of the ray through a parallel to the given ray; then any two planes Q, R through this line will cut P in lines which pass through the point of intersection, or the shadow of a. Find the projections of these lines, viewed as the intersection of P with Q and R respectively: then the intersections of their projections on the coordinate planes will be the projections of the shadow sought.

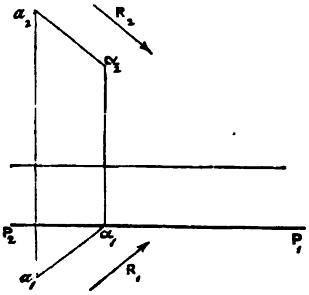
2. When the light emanates from a given point, the only difference will be that the shadow line instead of being parallel to R′ will pass through the projections of this point.

CASES OF PROP. V.

It will often happen that, under particular conditions of the data, the practical construction becomes much simplified; and the simplified cases which occur in usual practice require to be specially noticed.

1. Let the plane be parallel to the *elevation*, and P_1 be its trace, $a_1\,a_2$ being the given point, and $R_1\,R_2$ the direction of the ray.

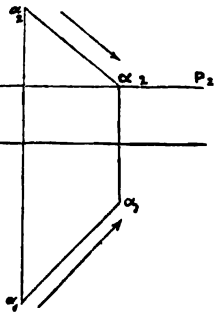
Draw the parallels through a_1 and a_2 to the projections of the ray; and let $a_1\,\alpha_1$ cut P_1 in α_1: then α_1 is the horizontal projection of the shadow point on P_1, and drawing, as usual, we get α_2 for the projection on the vertical.

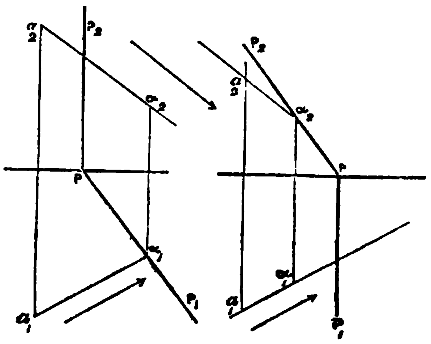


2. Let the plane be parallel to the *plan*, and P_2 be its trace, $a_1\,a_2$ as before. Then by merely reversing the order of the last operations we get α_1, α_2 for the horizontal and vertical projections of the point of shadow.

3. Let the plane be perpendicular to one of the coordinate planes, but not parallel to the other; as P in the two annexed figures.

Draw $a_1\,\alpha_1$ or $a_2\,\alpha_2$ (as the case may be) to meet PP_1 or PP_2, in α_1 or α_2: and proceed, as in the preceding cases, to find α_2 or α_1: then $\alpha_1\,\alpha_2$ is the point required.

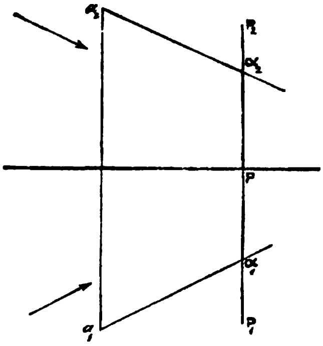




4. Let the plane be perpendicular to both planes: that is profile.

Draw $a_1\,\alpha_1$, and $a_2\,\alpha_2$ to meet the traces (parallel to the ray as before): then $\alpha_1\,\alpha_2$ is the shadow point required.

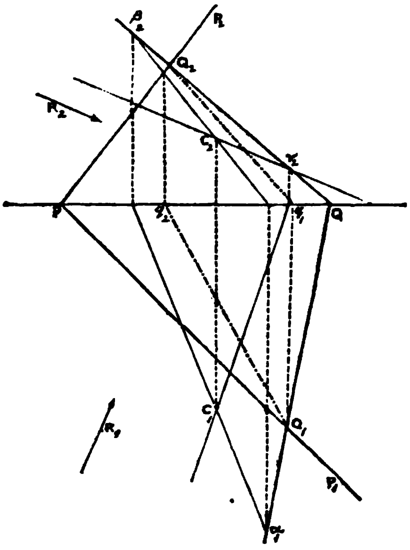
For all these operations are only finding the intersection of the ray with the given plane, by the methods of Descriptive Geometry, as already explained and studied.



PROPOSITION VI.

To find the projections of the shadow of a given line on a given plane.

Let P be the given plane, $\alpha_1\,\beta_2$ the given line, and R the direction of the rays.



1. Through any point c of the given line, draw a parallel to the ray R: then the plane of shadow of the given line will pass through this line and the given line; and its intersection with P will be the shadow on P.

Find one of the traces of the line through c parallel to the ray, as γ_2; then trace the plane $\beta_2\,\gamma_2\,\alpha_1$ which will be the shadow plane. Find its intersection with P: then the lines $Q_1\,q_1$ and $Q_2\,q_2$ will be the projections of the shadow cast upon the plane P by the line $\alpha_1\,\beta_2$.

2. If the light emanate from a point, the difference of construction

will merely be that which results from drawing the line through *c* from the luminary instead of parallel to R.

The cases of this proposition are correspondent to those of the preceding one, the operation being performed for each extremity of the given line.

LIGHT AND SHADE.

We shall omit, for the moment, all consideration of the circumstances arising out of the supposed *magnitude* of the luminous body, and therefore all that relates to penumbræ and penumbrals. The light will, therefore, be considered to either move in parallel straight lines, or in lines meeting in a single point. The influence of magnitude will be considered hereafter.

Only the actual effect of *direct light* as distinguished from its *absence* will be considered in this stage of the inquiry: all that relates to reflected and refracted light upon the parts of the body not in direct light, and all that relates to the different intensities of light upon the enlightened as well as upon the shaded parts, involving other and separate considerations, will not be discussed at present.

We shall discriminate the shade as here considered by the name of *geometrical shade*, and the shade modified by the other circumstances referred to, by the name of *natural shade.*

All figures bounded by plane surfaces have these surfaces bounded by straight lines meeting, three or more, in points. The boundary of light and shade must, evidently, be in some of these lines, and hence must also pass through some of these points. It will hence follow that the portion of space from which the light is intercepted by a polyhedron is a prism or a pyramid, according as the rays are parallel or emanate from a point. It is moreover clear that the *extreme* plane boundaries of the prism or pyramid will be the boundaries of light and shade; and that the parts (or faces) which lie on the side of the polyhedron on the side of the light will be enlightened, whilst the remaining ones are in geometrical shade.

When a plane face *coincides with* one of the planes of light and shade, we shall call it *neutral*:* the slightest variation of its position in one way placing it in light, and in the opposite way shade.

If the surface be *curved*, the boundary of light and shade will be a cylinder or cone, according as the rays are parallel or emanate from a point; and in all cases the cylinder or cone will be *tangential* to the surface, and may be considered as generated by the consecutive intersections of tangent planes to the surface drawn parallel to the given ray, or emanating from the given point, as the case may be.

To find the shaded parts of a polyhedron, which are always bounded by straight lines, we have only to find, therefore, the *extreme shadows;* and mark the lines whose shadows these are, which will be the lines of separation sought.

In the case of the cone and cylinder, the determination of the shadow also suffices for the determination of the shade, inasmuch as the generatrices which limit the shadow also limit the shade.

For other curve surfaces, as the surfaces of the second order, developable surfaces, and surfaces of revolution generally, a different mode of actual construction must be employed adapted to the known properties of each of these classes of surfaces; but the methods themselves are both rigorous and strictly geometrical.

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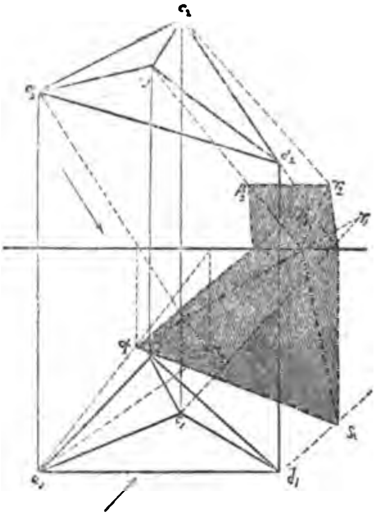
* It might perhaps be as well to call the lines of separation of light and *neutral lines;* or in polyhedrons, *neutral edges.*

PROPOSITION VII.

Given the plan and elevation of a polyhedron, to find the shadows of the figure on the plan and elevation, the light being parallel to a given line.

Find the shadows α, β, γ, δ of the angular points on whichever plane they may lie; and draw lines to join them if on the same plane; or if on different planes, find the broken line of shadow as before explained. These will be the boundaries of the shadow.

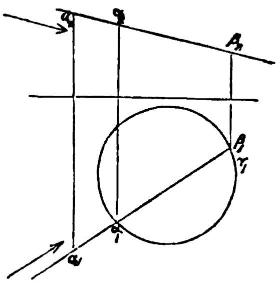


In the tetrahedron in the figure the shadow of *ad* is wholly on the horizontal plane, and of *cb* wholly on the vertical; whilst those of all the other lines are partly on each plane.

PROPOSITION VIII.

To find the shadow of a given point on a given right cylinder.

This is, in fact, a particular case of the intersection of a given line and a given cylinder, which has been constructed in the "Descriptive Geometry," (*Prop.* VII. *Sec.* IV.): but in this case, like many others in the application of general principles, the actual problem is *almost invariably* to the right circular cylinder, perpendicular to the plan. Our solution merely requires that the cylinder shall have its axis perpendicular to the plan.



Let $a_1\, a_2$ be the given point, and lett he curve $\alpha_1\, \beta_1\, \gamma_1$ be the trace of the cylinder on the plan; draw $a_1\, \alpha_1$ parallel to R_1 to meet the trace in α_1, and $a_2\, \alpha_2$ parallel to R_2, to meet the perpendicular from α_1 in α_2: then $\alpha_1\, \alpha_2$ is the projected point of section.

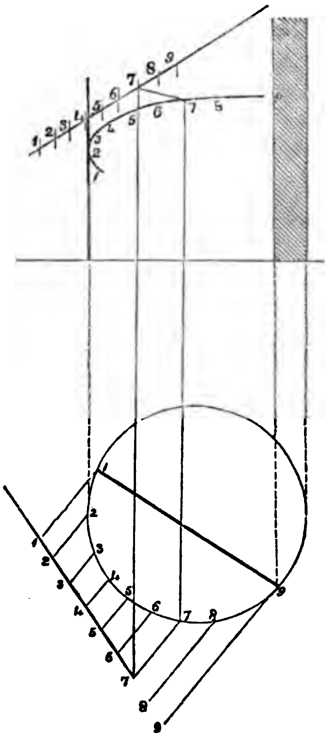
If $a_1\, \alpha_1$ meet the trace again in β_1 and we find β_2 as we found α_2, we get the shadow upon the internal face of the cylinder, the half next the light being supposed to be removed.

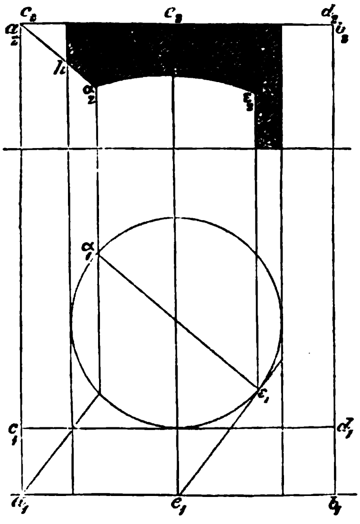
In this we have in fact constructed the point on the tangent plane of the cylinder, by the preceding propositions; and it follows from that without further reasonings.

Cases and Applications of Proposition VIII.

1. *To find the shadow of a line anyhow situated on the cylinder.*

This is effected by the construction of the points 1, 2, . . .*n* taken at any convenient distances from each other; either equal or otherwise.



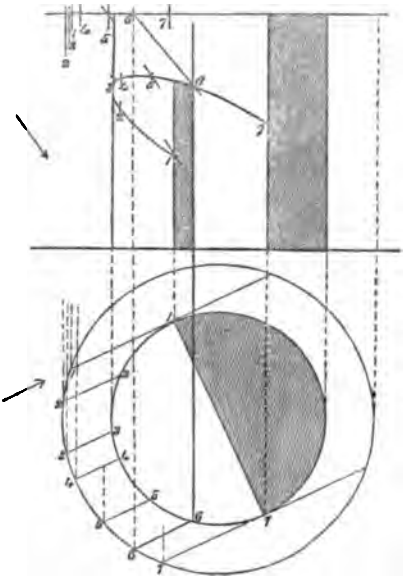


In the left figure, the line is *anyhow* situated: the construction for the point 7 being exhibited. The vertical edges of the cylinder 1 and

9 are the boundaries of light and shade, and 1, 9 on the plane is the trace of the neutral diametral plane.

In the right-hand figure, we have a cylinder surmounted by a square cap; $\varepsilon_1 \varepsilon_2$ is the line of shade; $\alpha_2 \varepsilon_2$ is the visible shadow of *ae;* and $\alpha_2 h$ is the shadow of *ac*, which, being perpendicular to the elevation, is a *straight line.*

2. *To construct the shadow of a circular cap surmounting a cylindrical column upon the shaft of the column.*



The process is here of precisely the same character; the construction being given for the point 6 of the cap.

And in precisely the same way will the construction be performed for any shadow whatever, the projections of the object being given.

3. *To find the shadow of a given circle on a given plane.*

(*a.*) Let the circle be parallel to the plane on which it is to be projected: then the shadow will be equal to the circle itself, and the homologous lines of the circle, and its projection will be equal and parallel, both in fact and in their respective plans aud sections.

(*b.*) When the plane of projection is one of the coordinate planes, we have only to find the projection of the centre, and with the radius of the given circle, describe the contour of the shadow.

(*c.*) When the plane has any given position, and the projection of the centre upon one of the coordinate planes is given, as well as that of any one point in the circumference, the other projections can be found as taught in the Descriptive Geometry, and the diameters parallel to and perpendicular to one of the traces of the containing plane can be projected on both coordinate planes. These diameters are perpendicular to each other in the given circle; and their projections are conjugate diameters of the elliptic shadow.

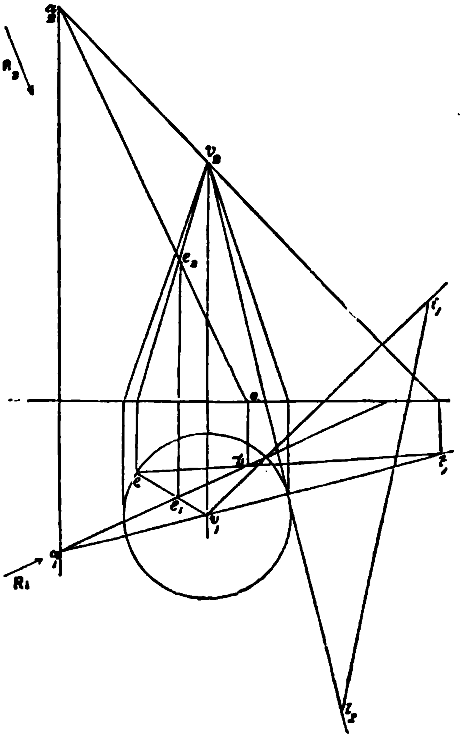
From these the principal axis can be found by several methods, and the ellipse traced.

(*d.*) When the plane which contains the circle is perpendicular to one of the planes, the actual work is slightly reduced, as far as relates to finding the vertical projections of the two rectangular diameters of the circle.

PROPOSITION IX.

To find the shadow of a point on a right cone standing on the plane.

Let $a_1\, a_2$ be the given point, $R_1\, R_2$ the given ray, and $v_1\, v_2$ the vertex of the cone. Through a_2 draw $a_2\, c$ parallel to R_2, and through a_1 draw the parallel to R_1, and eT_1 perpendicular to the axis, giving the point T_1. Find also the horizontal trace t_1 of $a\, v$: then $t_1\, T_1$ is the horizontal trace of the plane through av parallel to R.



Whence it will cut the trace of the cone in e, i, which are the horizontal traces of the edges of the cone on which the shadow falls externally and internally.

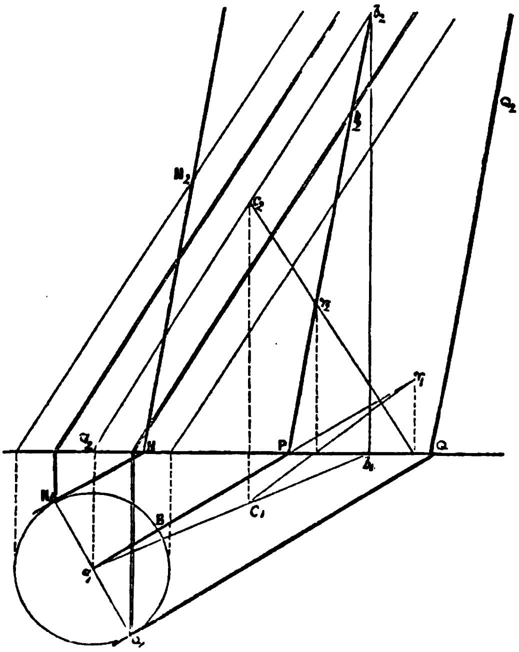
The intersection of the vertical and horizontal projections of these edges with those of the ray give $e_1\, e_2$ and $i_1\, i_2$ for the positions of the shadows externally and internally.

A moment's reflection will show that the process is precisely the same whether the cone be *right* or *oblique*.

PROPOSITION X.

To find the shade and shadow of a given cylinder.

Let $a_1 b_1$, $a_2 b_2$ be the projections of the axis of the cylinder, a_1, b_2 being its traces. Take any point c in ab and find the horizontal or vertical trace of the shadow cast by it, γ_1 or γ_2; then we get the plane P which contains the shadow of the axis.



The neutral planes are parallel to this, and their traces are tangents to the trace of the cylinder; and hence are found to be $N_1 N$ and $Q_1 Q$: also the vertical traces are parallel to PP_2, and N, Q are found; and hence NN_2 and QQ_2 are the boundaries of the shadow on the vertical plane.

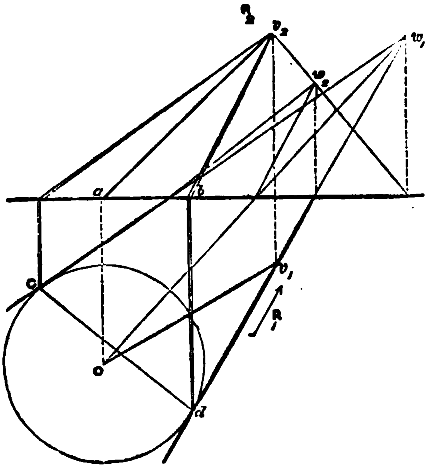
When the cylinder cuts the vertical plane, the visible shadow terminates; and the curve which bounds the shadow coincides with the trace of the cylinder on the vertical plane. This is easily marked by finding the intersection of the shadow of an edge and the projection of the same edge on the vertical plane.

For the shade we have the part of the cylinder on the right of $N_1 Q_1$ (in this figure, the light coming in on the left), and its corresponding vertical projection (bounded by *double* lines in the figure) to represent it.

PROPOSITION XI.

To find the shadow and the shade of a given cone.

Through v_1v_2, the vertex of the cone, draw a line parallel to the given ray: this will be the shadow line of the vertex of the cone. Let its traces be ω_1, ω_2.



Now, the shadow planes pass through this line and touch the cone itself: whence their traces upon the horizontal plane will be tangents to the trace of the cone.

Whence drawing tangents from ω_1 to the base of the cone, we have the traces of the two neutral edges, and hence can effect their constructions in the usual manner, as represented by the double lines in the figure.

Again, the horizontal shadows tend to ω_1, and from their sections with the axis, the vertical tend to ω_2. The shade and shadow are hence determined.

Note.—There is no practical simplification either in respect of the cone or cylinder, when they are vertical, except that the traces of the shadow on the vertical plane do not exist: the cylinder in this case not meeting that plane, and the cone not meeting it within the visible limits.

PROPOSITION XII.

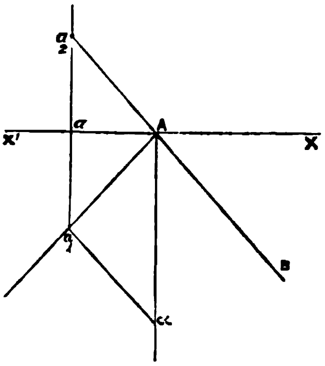
Given the azimuth and altitude of the sun to find the directions of the ray on the coordinate planes.

Let XX′ be the axis of the coordinate planes, AB the direction of the meridian with respect to it: make BAa_1 equal to the given azimuth, and make the angle $a_1A\alpha$ the altitude (on either side of Aa_1 as may be convenient), and draw from any point a_1 the line $a_1\alpha$ perpendicular to Aa_1: then lastly in the perpendicular a_1aa_2 to the axis make aa_2

$= a_1 a$ and join a_2A. Then will Aa_1 and Aa_2 be the parallels of the ray upon the coordinate planes.

For evidently Aa_1 is the horizontal projection by the nature of the conditions; and that aa_2 is the projection on the vertical plane of the projecting line of a point situated in the ray; and therefore Aa_2 is the vertical projection of the ray itself.

NOTE.—This, in fact, is finding the projections of that particular ray which passes through the intersection of the meridian and elevation trace on the horizontal plane.



It is also obvious how to construct the converse problem, should it be required, viz.:—

Given the projections of the ray to find the angle which it makes with *either* plane.

TOPOGRAPHIC PROJECTION,* OR THE METHOD OF CONTOURS.

SECTION I.

GENERAL PRINCIPLES.

1. It has been established in our preliminary course, that every point, every line, and every curve becomes perfectly defined, when its horizontal and vertical projections are exhibited in orthograph. Planes, it has been seen, are otherwise represented, viz., by means of their *traces* on the planes of projection. It has also been seen that the simple surfaces upon which we have treated (the cylinder, the cone, and the sphere), are also represented upon a different principle, that of representing an arbitrarily selected specimen of the line or circle by which they are respectively generated. Let us recal these.

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* The Topographic Projection differs from the old Orthographic Projection on a single plane, in giving the *scale* of the line or plane in addition to the projection. In the topographic the information is *completely* given; in the orthographic only partially. Or, at least, the complementary information is given in such a diversity of ways as to destroy all systematic representation. The consequential processes in the topographic method, hence, have a uniformity which cannot be given to those in the orthographic. Nicholson, indeed, attempted to remove this objection to the orthographic projection by means of his "directing diagram" (nearly 40 years ago in Rees's Cyclopædia, and more recently in his work on Projection), but with limited success.

2. A cylinder is generated by a line, which is always parallel to a given line, and which has its horizontal or vertical trace in a given curve.

3. A cone is generated by a line which always passes through a given point, and has its horizontal or vertical trace in a given curve.

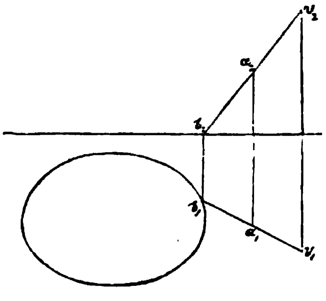
4. A sphere is generated by the revolution of a given semicircle about a diameter which is given in position.

5. Although we have employed the generating line and the generating circle respectively for the ordinary constructions of Descriptive Geometry, in representing these surfaces, these are not the *only* views under which we may consider these surfaces generated. In fact, when we come to consider surfaces generally in reference to the practical applications of Descriptive Geometry to the arts of construction, we shall seldom find these definitions to be the best adapted to our purpose, and hence I have dwelt upon them under this view but very slightly. Even this slight view of them under such an aspect has been adopted more for the sake of an easy transition from one view to the other than for any other reason; though it must he remarked, that even these afford facilities in certain special cases for the investigation of particular problems that are greatly simpler than the general mode of construction that is usually employed; this is, however, only casual, and is no argument in favour of its supplanting generally the more effective method which will be hereafter developed.

6. The most simple of general methods of representation of surfaces for the purposes of plan-drawing is that of supposing the surface, however generated, to be cut by a series of planes parallel to one of the planes of projection; and the curve of section to be represented by its projections on the two planes. The sections are in practice made, most commonly, by planes parallel to the horizontal plane of projection: though, as in Descriptive Geometry, there is no preference of one over the other in theory, and the horizontal is chosen from its being more in accordance with our earlier practice in this class of drawings.

7. For the purpose of showing this more clearly, I shall resume the representation of a cone as already given, and as given by the means here spoken of.

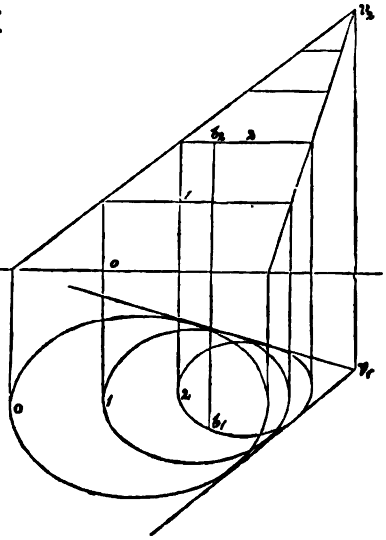
Let v_1 v_2 be the vertex, and a_1 a_2 a point in the surface: then, if the point b_1 be in the trace of the cone on the horizontal plane of projection, any point in the surface is represented if we join the point b_1 in the trace, and either of the projections a_1 or a_2 of the point through which the generatrix passes. The construction, and therefore exhibition of the defining elements, of any point in the conical surface, can be effected, provided we have the requisite data for fixing that point.



8. If, instead of the preceding representation, we employ the method of horizontal sections, we shall have a series of curves 0, 1, 2, etc., upon the horizontal plane of projection similar to one another; whilst

the vertical projections are horizontal lines bounded by the lines which are the projections of the extreme generatrices, and which are also marked in corresponding numbers 0, 1, 2, etc.

It will be obvious that any point *b* on the surface of the cone will be as distinctly defined, by its projection on one of these planes, as by the other method. It will be seen that whether b_1 or b_2 be given, the other can, by means of known processes, be found. A very rational inquiry in this case will be, whether, since the vertical plane contains only straight lines, the constructions effected by means of them might not, possibly, be more simply performed without their aid by the substitution of other operations whose results shall be coeffective with them. In many cases it will be found to be possible, but not in all cases. The principal *practical* advantage is in those cases where the projections on the vertical plane are upon a small scale in comparison with the extent of those on the horizontal plane, or *vice versâ*. To the engineer it occurs in military drawings, where the relief is small in comparison with the ground fortified; and to the civil engineer, in the delineation of ground which varies inconsiderably from the horizontal level. To him it is of importance for the purposes of railroad surveys, those of canals and turnpikes, the disposition of buildings, sewerage, supply of water, and parochial maps.



9. To show the general principle, let us suppose these horizontal sections to be made at distances from each other equal to any given unit of altitude. Then the horizontal projections, being drawn and numbered 0, 1, 2, etc., will suffice for the adequate representation of the sections themselves. The vertical projections may be omitted from the drawing, so long as our object is merely to enable the mind to form a general conception of the form of the surface; as it is easy to connect together the ideas of altitude and form—it being, in fact, nothing more than conceiving the several projections of the section (which are equal to them in all respects) to be raised up to the heights 1, 2, 3, etc., designated by the number of units affixed to the curve of projection on the horizontal plane.

10. In actual surveys, however, our object is an inverted one. The form of the surface is not given, but is required to be found by observation in each case. It cannot in general be a regular surface, that is, one whose geometrical genesis can be assigned. The *surveys* for this purpose will be explained elsewhere: and I shall assume that they can be made in what I am now discussing. The problem, as one of surveying, is:

On any given azimuth (or compass-bearing) to find the horizontal distances of all the successive pairs of points which have a difference of altitude of a foot.

It is not always possible, from the interference of obstacles, either to vision or motion, to obtain such observations by means of *direct levelling;* but when this is not possible, the survey can be so arranged that, when laid down for an adequate number of properly selected observations, these points can be assigned by numerical or graphic interpolation, whichever we choose to employ.

11. If we suppose a sufficient number of these points to be found and plotted on the drawing, a curve line may be traced by hand and humoured to suit our memory of the ground, through all of those marked of the same altitude; this will be a representation of that horizontal section of the ground which is at the assigned altitude from the plane to which the observations had reference. This line is technically called a *contour.*

This method of delineation has long been in use for marking the soundings or depths of water on coasts. It led, therefore, to the adoption of the plane of mean low water as that from which depths were estimated, and of course from which the heights of the adjacent coasts were likewise estimated. Still, though so long practised in such coast-charts, its principles and convenience of application were not perceived by surveyors and topographers till a recent date. It was, indeed, proposed (the suggestion being from a different source) a century ago to the French *Academie;* but it was only subsequent to the general peace that it was adopted by any public body. In 1818 the Bureau de Cadastre laid it down as the plan upon which the general topographical survey of France should be delineated; and in 1838, it was adopted in the Ordnance Survey of Ireland. Amongst the French scientific writers on projection, I have found only two who have given even a sketch of it—Leroy and Olivier; and in English, not one who has given even an intelligible description of the process, either as a mental theory or a practical operation.

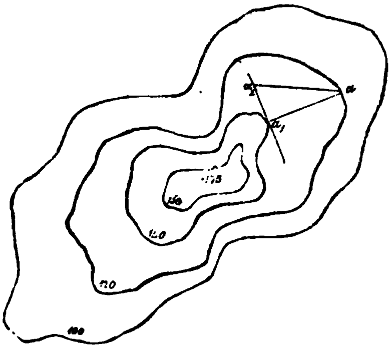
12. The plane of mean low water is not, however, found to be a convenient one for inland surveys, on account of the great difficulty of finding the real height above that plane at any one point in the survey. For military plans a plane *above* the highest point of ground is often used, as that to which the points of the survey shall be referred; and the system is estimated by depths *below this plane,* instead of *heights above* a horizontal plane taken on the ground. The one or the other may be used in our reasonings, inasmuch as when the distance of the two supposed planes is known, the distance of a point from one of them being given, its distance from the other is found by simply subtracting the given distance from the distance of the two planes.

13. The plane to which the system is referred is called the *plane of comparison:* and the reference of a system of points given in respect to one plane, to another given plane, is called *changing the plane of comparison.*

14. As it will preserve a closer analogy to the projections with which we are most familiar, we shall in our investigations refer the system to a horizontal plane *below* the system itself.

15. Let us suppose, then, that from actual surveys we have obtained

the following lines of 100, 120, 140, 160, and the highest point of 175 feet above the plane of comparison. It will appear at once that the nearer these lines approach to each other, the more steep the ground is in that region, and the more distant the less steep.



16. It will also be apparent that if at any point a_1 the declivity be required, we have only to draw a perpendicular to the curve at a_1 (or to its tangent), and making $a_1\, a_2$ perpendicular to $a_1\, a$, and equal to 20 feet of the scale, and joining $a_2\, a$, the angle $a_1\, a\, a_2$ is the elevation of the hill-side at a; or which comes to the same thing, its declivity (or depression below the horizon) at the point of which a_1 is the horizontal projection.

17. When the declivity is considerable, or the lines of contour numerous, the projections become inconveniently close to each other. In this case it is usual to measure the vertical and horizontal distances by different scales, the horizontal being 10 or 100 times that of the vertical. This, though it alters the appearance to the eye of the general declivity of the ground, renders the constructions which are often necessary to be made on such a map for engineering purposes more distinct and consequently more certain. We shall only require a final *transformation of the results*, which is easily understood and readily made.

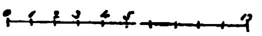
18. In the same manner, if the declivity be very small, and both horizontal and vertical measures be made on the same scale, the contours will expand themselves to an inconvenient extent. This is remedied by taking the horizontal scale $\frac{1}{10}$ or $\frac{1}{100}$ of the vertical one: for though to the eye it conveys the notion (if unapprised) of greater than the actual declivity of the ground, its operations are more easily effected than otherwise. The same remark applies to it as to the preceding case, respecting the final result.

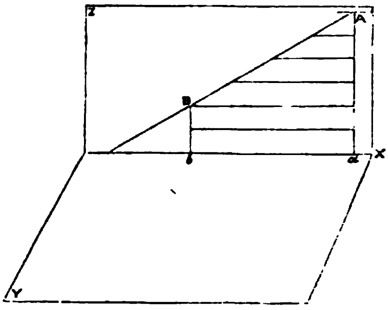
19. When both scales are the same, we shall call it the *natural system of contours*: when the horizontal scale is diminished, we shall call it the *compressed system of contours*: and finally, when it is enlarged, the *expanded system of contours*.

SECTION II.

THE GRADUATION OF HORIZONTALLY PROJECTED LINES.

1. Let the line here graduated be that by which altitudes above XY are laid down; and let AB be a segment of a line, the plane AB *ba*

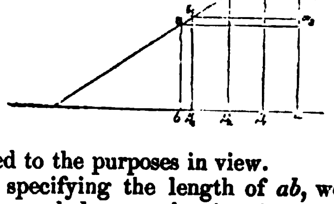




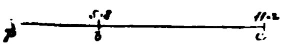
its projecting plane on the horizontal, and *ab* its actual projection. Then A*a* and B*b* the altitudes above the plane XY, are assumed as being measured or deduced from actual survey, and likewise the position of *a* and *b*. Then, obviously the line AB is given in all respects; and any required particular can be obtained either by construction or calculation.

It will not be necessary to actually construct the line in its real position for any of the practical uses to be made of it, though for the purposes of reasoning we must sometimes recur to the eidograph and turn the system upon the horizontal plane.

If, in the *natural system*, Aa_1 be a unit of the altitude scale, then $a_1 b_1$ or $a\beta_1$ will be the unit of the projection scale corresponding to it, and Ab_1 a unit of the declivity-scale, also corresponding to Aa_1. Instead of using the same scale-unit for all the three lines, this method employs the same number of units in designating any three corresponding segments of the lines. Our first business is, the graduation of such lines adapted to the purposes in view.



Suppose that, without numerically specifying the length of *ab*, we merely represent it on any given scale, as below; and write the numbers 11·2 and 5·8, which designate the altitudes at *a* and *b* of points A, B, in the line: then this is considered to represent the line and points A, B, in it—these numbers being either referred to the fore-mentioned scale, or to another which has a specified relation to it.



The relation of the scales being given, together with the position of *a* and *b*, with their respective numbers 11·2 and 5·8; the line AB is said to be *figured*: and these numbers are called the *figures of the line;* and *a*, *b*, the *projections of the figures.*

2. In our general investigations we shall view them as referred to an arbitrary point *p* in *ab*, as the origin of abscissas, and the altitudes as their corresponding ordinates. We shall thus, denoting A by $x_1 y_1$, B by $x_2 y_2$, any other point by xy, and the inclination by θ, obtain at once the following relations:—

$$(y - y_1)(x_1 - x_2) = (x - x_1)(y_1 - y_2) \quad . \quad . \quad . \quad . \quad (1),$$

$$(y - y_2)(x_2 - x_1) = (x - x_2)(y_2 - y_1) \quad . \quad . \quad . \quad . \quad (2),$$

$$\tan\theta = \frac{y_1 - y_2}{x_1 - x_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

The two first of these are identical, and either may be used. From either of them x or y may be found when the other five are given. The third gives the inclination of the figured line to the plane of comparison.

Again, if we make $y = o$, in either the first or second equation, we obtain the value of x, which corresponds to the trace of the line upon the plane of comparison. It is

$$x = \frac{y_1 x_2 - y_2 x_1}{y_1 - y_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

and giving to x this value, we obtain the *zero point* of the projected line.

Let u be a unit on the vertical scale: then on the projection and on the line itself the corresponding units will be respectively,

$$u_p = u \cot\theta; \text{ and } u_l = u \operatorname{cosec}\theta \quad . \quad . \quad . \quad (5).$$

If, however, the scale be compressed or expanded n times, the projective unit will respectively be

$$u_p = \frac{u \cot\theta}{n}; \text{ or } u_p = n . u \cot\theta \quad . \quad . \quad . \quad . \quad (6);$$

and the corresponding units on the line itself

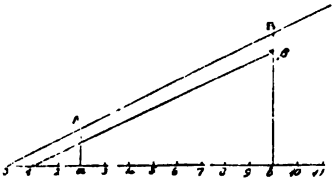
$$\left.\begin{array}{l} u_l = \dfrac{u}{n} \operatorname{cosec}\theta, \text{ and} \\ u_l = nu \operatorname{cosec}\theta. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The units on the line itself occur less frequently than in the other case, in actual researches.

One or two examples adapted to numerical data will suffice to show the use of these formulæ.

3. We shall now proceed to show the constructions which are involved in the use of this mode of viewing a straight line by means of its projection and figuring.

Denote by *a′*, *b′* the figuring of the points, *a*, *b*. Set off *a*A, *b*B perpendicular to *ab* from the selected scale; and draw BA to meet *ab* in 0. Then 0 is the zero point of the line AB, as represented on its projection *ab*.



To graduate it, make B*β* equal to the vertical unit, and draw *β*1 parallel to AB; then 01 is the projecting unit. Step this distance from 1 to 2, from 2 to 3, etc.; then we have the scale to which we must recur in all that relates to this line, and to all others in the same drawing which have the same inclination to the plane of comparison.

If the vertical scale be increased (suppose tenfold, as is usual in delineations of ground of slight inclination), then these divisions 01, 12, etc., will be, not units of projection, but as many units as the vertical scale is increased. Thus, if Bβ were 10 vertical units, 01 would require sub-division into 10 parts, and so on. Also, if the drawing were expanded to represent very steep ground, the units of projection would be enlarged in the same ratio.

The line so graduated is called the *scale of the line* AB; and it is the first step to be performed in any problem in which the line is concerned.

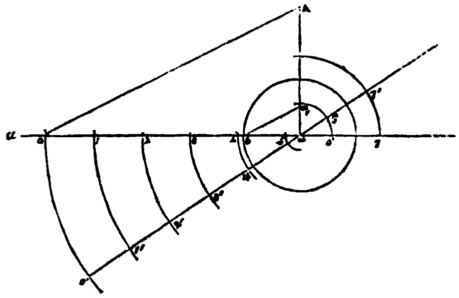
It must be carefully recollected, however, that all the lines in the same drawing must be referred to the same vertical unit. It will, however, be obvious that no difference of constructive process results from using one scale rather than another; but a difference in the ultimate estimation of the values of the quæsita will be necessary, and a difference in the *visual* result will also ensue. In the following problems the *natural system* will be used in general; and where otherwise, it will be specially notified.

PROPOSITION I.

Through a given point, to draw a line making a given angle with the plane of comparison, and to graduate it.

This problem is evidently indeterminate in one respect, viz., that such lines will radiate in all directions round the given projection; but whichever be selected, the process will be the same; and if found for one, may be transferred to all the others by concentric circles, whose centre is the projection of the given point.

Let a be the projection; draw any line $a\alpha$; perpendicular to which



make aA the scale length of the altitude of the given point A; and in it take aa_1, a unit of that scale; and make ab the cotangent of the given angle to that unit; join a_1b and parallel to it draw A0. Then 0 is the zero point of the line required; and if from 0, the distance ab be set off to 1, 2, 3, . . . successively, the projection of the line is constructed and graduated.

If aa_1 be of a different scale, either greater or less than the horizontal one, then these divisions 01, 12, etc., will represent only parts of the projection corresponding to that relation. If aa_1 be a multiple of the vertical unit, 01 will be the same multiple of the projective unit; and similarly, if aa_1 be only a part of the vertical unit, 01 will be the same part of the projective unit.

Moreover, it is clear that $0'a$ is also one of the lines which fulfils the conditions of the problem.

PROPOSITION II.

Having given two figured lines, and a point in one of them, to draw the projection of a horizontal line through the given point to meet the other line.

Let *ab* be a given line, and *p* a point in it; to draw the projection of a horizontal line through *p* to meet another line *cd*.

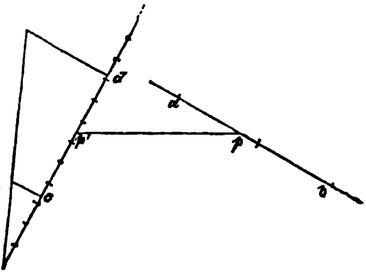


Figure the line *cd* as before directed, and let *p'* be the point in it which is figured with the number at *p*. Then draw *pp'*; it will be the projection required.

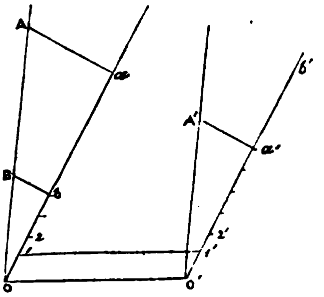
For since *p* and *p'* are at the same distance from the plane of comparison, they are points in a line parallel to that plane.

In a manner precisely similar may any number of lines be drawn through as many given points; in whichever of the given lines the points be situated: or again, whether the points be in a specific line or not.

PROPOSITION III.

Through a given point to draw a line parallel to a given line.

Let *ab* be the projection of the given line figured at *a* and *b*, and let *a'* be the projection of the given point, also figured.



Now, since the lines are to be parallel, their projections will be parallel, and hence the *position* of the projection *a' b'* through *a'* is given.

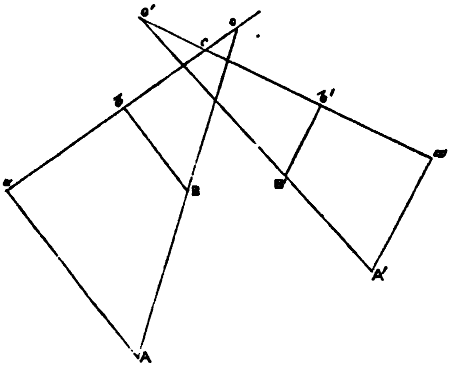
Make *a* A, *b* B, *a'* A' perpendicular to *ab*, *a' b'*, and equal to the given figures. Draw AB to meet *ab* in 0: then 0 is the zero point of the given line. Through A' draw A' 0' parallel to A0: then 0' is the zero point of the line sought. Draw 11', 22' . . . parallel to 00': then the projection *a' b'* is graduated.

PROPOSITION IV.

Two lines are given, to find whether they intersect, and if so, at what figure.

(1.) If their projections be parallel, they can never meet, whatever their scales may be, for they are then in parallel planes.

(2.) Let their projections intersect (viz. *ab* and *a′ b′*) at *c*. Graduate them; then if *c* represent the same number on both lines, each in reference to its own scale, the lines meet in the point which is projected in *c*; and the figure of that point is the altitude of their point of intersection.



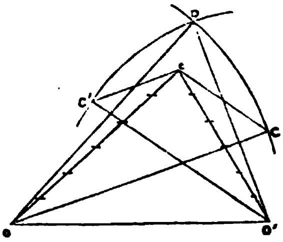
If *c* upon the two lines do not represent the same altitude, the lines do not meet; and the difference of these numbers is the difference of the altitudes of the points in the two lines which have a common projection.

PROPOSITION V.

Given the figured projections of two lines which meet, to find the eidographic inclination of these lines.

Let *c* be the intersection of the projections, and 0, 0 the zero points of the given lines. Make *c*C, *c*C′ perpendicular to 0*c* and 0′c respectively, and each equal to the figure of the point *c*. From centres 0, 0′ with radii 0C, 0′C′, describe arcs intersecting in D; and join 0D, 0′D: then 0D0′ is the eidographic angle of the given lines.

For it is evident that 0D0′ is the angle made upon the plane of projection by the two given lines after revolution.



Scholium. If the lines should have but slight declivity, the zero points may fall at an inconvenient distance. In this case any two points *having the same figure* may be taken instead of them, only remarking that the perpendiculars at *c* in the preceding construction must be diminished also in the same ratio.

For we are thus in reality only changing the plane of comparison and all the work remains, like the reasoning, precisely similar to that already employed.

SECTION III.

PLANES.

1. In this kind of projection a plane is defined by its trace on the plane of comparison, and its inclination to that plane.

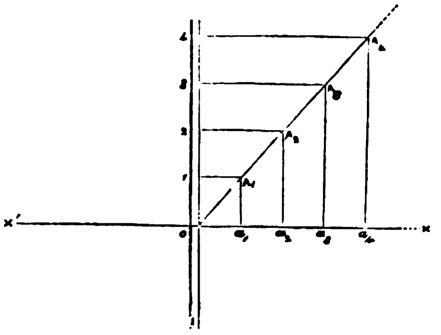
If we cut the trace of a given plane by a profile plane, that plane is perpendicular both to the given plane and to the plane of comparison; and the common sections of the profile plane with those which it cuts are two lines, one in each of the planes, that contain an angle equal to the inclination of the given plane to the plane of comparison (*Pls.* II., 18, *and Def.* 4). One of these lines is also the projection of the other; and by being properly figured, may represent the angle of inclination of the given plane to the plane of comparison. This is the mode of representation adopted for planes in this system; and this angle we shall express by *declivity*, or the declivity of the plane. One form of expression and construction founded on it will be—*a plane is defined by its trace and declivity.*

2. If lines be drawn in the given plane parallel to its trace, and at distances from that trace equal to the unit of the line of declivity (or at distances from the plane of comparison equal to the unit of altitude), their projections will be parallel to the trace, and at distances on the projection of the line of declivity equal to the unit of the general scale of projection in the system employed.

PROPOSITION I.

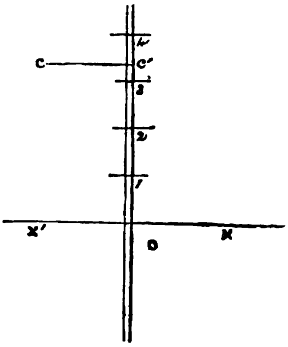
To form the scale for figuring a given plane.

Let X′X be the trace of the plane, and conceive a plane drawn perpendicular to it, cutting the given plane in a line and the plane of comparison in 04: about 04 let the profile plane be turned to coincide with the horizontal plane, so that the line of declivity shall take the position $0A_4$: draw the perpendiculars $A_1\,a_1$, $A_2\,a_2$, . . . to 0X: then $0a_1$, $0a_2$, $0a_3$ . . . are equal to the altitudes of the points in the line of declivity above the plane of comparison.



Wherefore we have this construction:—

Draw (first fig.) a line 04 . . . perpendicular to the trace X′X of the given plane: make the angle $40A_4$ equal to the given angle of declivity: make $0a_1$, $0a_2$, . . . a scale of altitudes, and draw a_1A_1, a_2A_2, . . . parallel to 04: then, lastly, draw $A_1 1$, $A_2 2$, $A_3 3$, . . . parallel to the trace. The scale of the plane is then graduated in 0, 1, 2, . . .



In practice it will be easier to find merely a_1A_1, and step it off from 0 along 04.

Cor. We may readily find whether a given figured point be in a given figured plane or not. Let *c* be the projection of the point (second fig.): draw *cc′* parallel to the trace X′X, and the point in question will be in the plane if 0*c′*, be the figure in the given point: otherwise not.

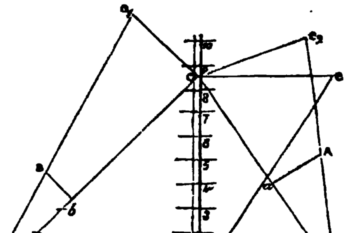
Scholium 1. Should the trace be inconveniently situated for use, any line parallel to it may be used instead, if its altitude be given; always recollecting that the numbering must proceed both ways from that point in the scale.

Scholium 2. For the sake of distinguishing to the *eye* the graduations of a line and a plane, it is usual to mark the scale of a plane with a *double line*; and the one is drawn a little thicker than the other, for a reason which will presently appear.

PROPOSITION II.

To draw a plane through three given points.

Let *a*, *b*, *c*, be the projections of the three given points A, B, C. Construct the lines through CA, CB as before (*Sec.* II., 2), and let 0_1, 0_2 be their traces: join 0_10_2 and draw the perpendicular *c*0 to 0_10_2. From the graduations 1_2, 2_2, 3_2 . . . of one of the lines, draw parallels to 0_10_2, cutting *c*0 in 1, 2, 3, . . . we shall then have *c*0 graduated for the plane through the three points.



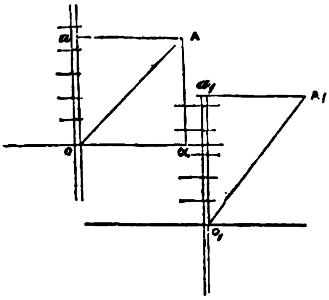
Cor. 1. The construction is evidently the same as that for a plane to pass through two given lines.

Cor. 2. If we wish to exhibit the inclination of this plane to the plane of comparison, we must draw *c*C parallel to $0_1 0_2$, and make it equal to the figure of the point *c*, and join C0. The angle *c*0C is that required.

For it is in fact the angle of the profile plane placed on the horizontal plane, and 0C is the line of declivity.

PROPOSITION III.

To draw a plane through a given point parallel to a given plane.



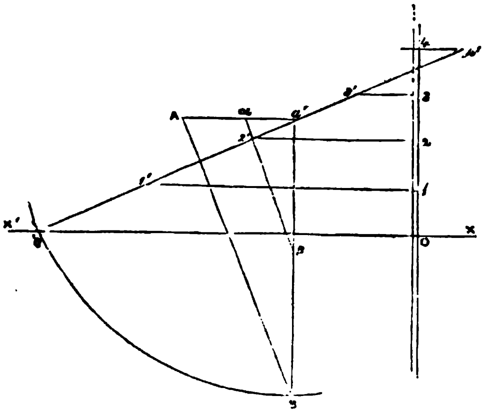
Let $0a$ be the graduation of the given plane, and a_1 the projection of the given point: draw the scale 0_1a_1 of the plane through a_1 parallel to $0a$: make 0α equal to the figure of the given point: draw 0A to make the declivity of the given plane with $0a$, meeting the perpendicular $A\alpha$ in A, and draw Aa parallel to 0α. Make $a_10_1 = a0$: then 0_1 is the zero point of the line of declivity of the plane sought.

Also, since the planes are parallel their traces are parallel, and the point 0_1 is given; therefore the trace and scale of the plane are found.

PROPOSITION IV.

Through a given point in a given plane to draw a line that will make a given angle with the plane of comparison.

Let a' be the projection of the given point; draw a'B perpendicular



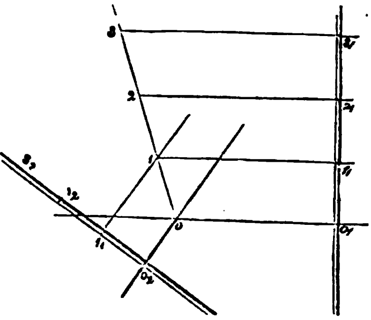
to X′X the trace of the given plane, and a'A parallel to it and equal to the figure of the given point: make $a'\alpha$ the unit of the vertical scale to which the given plane is figured, and draw $\alpha\beta$ to make the angle $\alpha\beta a'$ equal to the given angle with the plane of comparison, and draw AB parallel to $\alpha\beta$. With centre a' and radius a'B describe the arc B0′, cutting the axis in 0′. Then $0'a'$ is the projection of the line; and its figures are found by parallels 1′ . . . to XX.

For, if we consider that AB of this figure is a line which fulfils the condition; and that $0'a'$ is the projection of same line revolved about the projection a' till it comes to the given plane, the truth of the construction will be apparent.

PROPOSITION V.

To construct the intersection of two given planes.

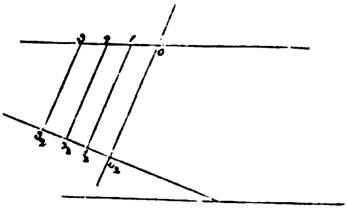
Let OO_1 and OO_2 be their traces, and the scales $0_1 3_1$, $0_2 3_2$ be figured: through 1_1, 1_2 draw parallels to the traces intersecting at 1: then 01 being drawn indefinitely, will be the projection of the line required, and in length equal to a unit of its figuring.



For $1\,1_1$ is the projection of a line parallel to the trace at the distance unit from the plane of comparison, and hence is the projection of all the points in the plane (1) which are at a unit's distance from the plane of comparison. In like manner $1_2 1$ is the projection of all the points in (2) which are at that distance from the plane of comparison. Whence their intersection 1 is the projection of the point common to both planes which is a unit distance from the plane of comparison.

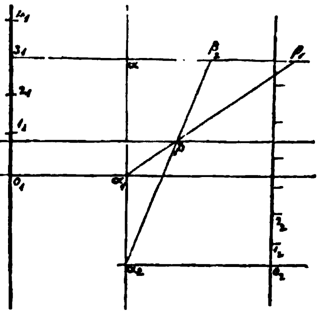
Also the projection of the intersection of the given planes is a straight line, and it passes through 0 and 1: and hence it may be figured from having its zero point, projection, and projected unit given.

COR. 1. If one of the planes, as (1), be perpendicular to the plane of comparison, its scale will disappear, and the plane will be represented by its trace only. In this case the trace of this plane is the actual projection of the line of intersection sought; and the scale of this line will be found by drawing lines $1_2 1$, $2_2 2$ . . . from the points of the scale of the other plane parallel to its trace, to meet the trace of the perpendicular plane.



Cor. 2. If the two planes have their traces parallel, their line of section will be parallel to them, and hence also its projection on the plane of comparison will be parallel to them both. The preceding method is therefore inapplicable to this case; but the following will obviate the difficulty:—

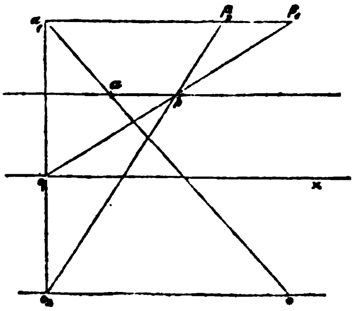
Let $0_1 4_1$, $0_2 4_2$ be the scales of the given planes: conceive the eidograph cut by a profile, meeting the traces in α_1, α_2: from any point (as 3_1) draw the parallel to the traces, and set off $\alpha\beta_2$, $\alpha\beta_1$, equal to the altitudes belonging to 3_1, upon the scales of the planes. Then if $\alpha_1\beta_1$, $\alpha_2\beta_2$ intersect in p, the parallel through p to the traces will be the projection required. Its figure is uniform in reference to either scale, though different, of course, in the two.



PROPOSITION VI.

To find the intersection of a given line with a given plane.

Let $a_1 0$ be the projection of the given line: construct the line of section of the given plane with a plane through the given line having its trace parallel to the given trace (*Prop.* v. *Cor.* 2). Then the point of section of the line and plane is projected at some point in this line. It is also in the projection $a_1 0$ of the given line, and hence at their intersection a. This may be figured as a point in the plane, or as a point in the line, according to the ulterior objects in view.

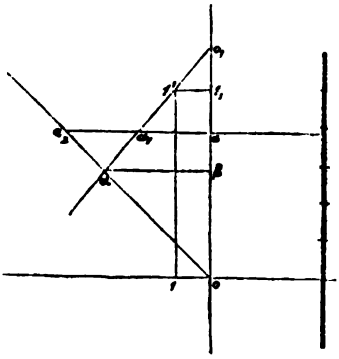


PROPOSITION VII.

To draw a line perpendicular to a given plane through a given point·

Let a be the projection of the given point, and conceive a profile plane through it, cutting the trace of the given plane in 0: draw aa_2 parallel to the trace; and in it take aa_1 the figure of the point, and aa_2 the figure of the scale of the plane corresponding to a. Join $a_2 0$, and draw $a_1 \alpha$ perpendicular to $0a_2$, meeting $0a$ in 0_1 and $0a_2$ in α. Make 0 1 the unit of altitude, draw $11'$ parallel to $0a$, and $1'1_1$ parallel to the trace. Also, draw $\alpha\beta$ parallel to 0 1.

Then 0_1 is the zero point of the perpendicular, $0_1 1_1$ is the unit of its projection, $a_1 \alpha$ is the length of the perpendicular, and β is the projection of its point of intersection with the given plane.



For since the line is perpendicular to the plane, its projection is perpendicular to the trace of the plane, which condition is in the first place fulfilled, since $\alpha\beta$ is perpendicular to the trace.

In the next place, the profile plane being turned about $0a$, the line of declivity becomes $0a_2$, the position of the point becomes a_1, and the perpendicular to the plane is perpendicular to $0a_2$. Whence 0_1 in which it meets $0a$ is the zero point, and the scale is obviously true. It also follows that we have $a_1 \alpha$ equal to the distance of the point from the plane; and β the projection of α is obviously the projection of the point of section.

Cor. The declivity of the line and plane being complementary of each other, the figuring may be effected without quite so much constructive work. The method of effecting it is left as an exercise.

PROPOSITION VIII.

Through a given point to draw a plane perpendicular to a given line.

The line being given, its projection and zero point are given: and the trace of the required plane will be perpendicular to that projection. Assume therefore a plane at pleasure, having its trace such as to fulfil that one condition; and figure this plane. Then through the given point draw a plane parallel to this (*Prop.* III.) and it is that required.

This, manifestly, is an exact converse of the preceding proposition.

PROPOSITION IX.

Through a given point to draw a plane which shall be parallel to two given lines.

Let 0_1, 0_2 be the zero points of the figured lines which are drawn through them, these being the given lines, and let a be the point through which the plane is to pass. Through 0_2 draw a line parallel to the other given line (*Prop.* III., *Sec.* 2.), and figure it to the same scale: then this will be a line parallel to (1). Hence the plane which contains these lines is parallel to the line (1), and therefore to the required plane. Join 1_2 1_1: then this is the projection of a line parallel to the trace of the required plane at a unit of altitude from the plane of comparison. Wherefore we can trace and figure the plane which contains these lines.

Through a draw a figure scale parallel to this, and from a set off the given figure of a_1 to 0: the line through 0 parallel to the former trace is the trace required; and this scale being figured, we have the plane determined.

PROPOSITION X.

To find the angle contained between two figured planes.

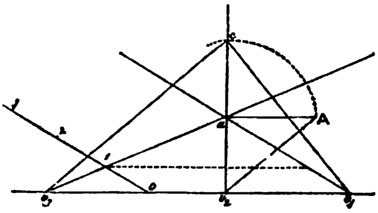
Let (1), (2) be the given figured planes (the diagram is easily conceived), and $0b$ the figured line in which they intersect (found as in *Prop.* V.): through any convenient point in $0b$ as 1, draw a line $1a$ at right angles to $0b$, and make it equal to the vertical figure of the point 1: join $0a$, and draw ab perpendicular to $0a$: through b draw cd perpendicular to $0b$ to meet the traces of the given planes in c, d, and in $b0$ make $ba = ba$: then drawing ca, ad, the angle cad is the inclination of the planes to one another.

For, as was shown in the Descriptive Geometry, cad is the triangle revolved on the plane of comparison in which the plane through 1 cuts the two given planes and the plane of comparison.

PROPOSITION XI.

There is given the trace of a plane, and a figured line in it; it is required to draw through a given point another line, which shall lie in the traced plane and make a given angle with the given line.

Let 0 3 be the figured line in the plane whose trace is $0_1\,0_2$, and let a be the projection of the given point: through a draw $a0_1$ parallel to 0 3, and figure it: draw $a0_2$ perpendicular to the trace of the plane, and make aA equal to the given figure of the given point: join 0_2A, and make $0_2c = 0_2$A, and join $0_1 c$: make the angle $0_1 c 0_3$ equal to the given angle and draw $0_3 a$, and figure it by parallels to the trace $0_3 0_1$. This is the projection sought.

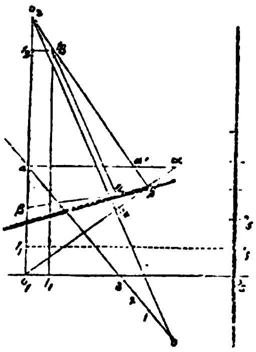


For, obviously, $0_1 c 0_3$ is equal to the triangle whose projection is $0_1 a 0_3$; and hence all the conditions as to magnitude are fulfilled.

PROPOSITION XII.

Through a given line to draw a plane perpendicular to a given plane.

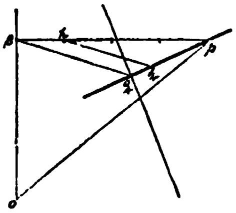
Let $0_5\,2_5$ be the scale of the given plane, its trace being $0_1\,0_5$; let a be the projection of any point, taken at pleasure, in the given line 0a; draw $a0_1$ perpendicular to $0_1\,0_5$, and $a\alpha$ parallel to it; make aa' the figure of the selected point, and $a\alpha$ the figure of the given plane for that point; join $\alpha 0_1$, and draw $a'p$ perpendicular to $0_1\,\alpha$, meeting $0_1 a$ in o_2. Then 0_2 is the trace of this perpendicular (*Prop.* VII.), and the required plane will have its trace passing through this trace 0_2. The trace of the required plane also passes through o the trace of the given line; and hence 00_2 is the trace of the plane required.



Draw $p\beta$ parallel to $0_1\,0_5$: then this is the figure of a point in the plane drawn. If we draw the scale line $p0_4$ perpendicular to the trace, and graduate this proportionally to $p\beta$, it will be the scale of the required plane, and the problem is solved.

For this graduation different methods may be employed, one of which is the following:—

Join $\beta 0_4$, and make βh a unit of altitude: then $h1_4$ parallel to $\beta 0_4$ meets the scale line in 1_4, the first division of the scale of the required plane.

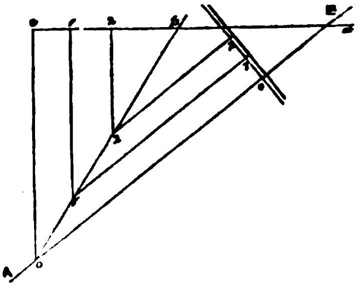


PROPOSITION XIII.

To find the intersection of a given line with a given plane.

Let 0*a* be the projection of the line, AB the trace of the plane, both figured 0, 1, 2.

Through 0 draw 00 perpendicular to 0*a*; then this is the trace of a plane through the given line whose declivity is the same as that of the given line; and hence whose scale is 0*a*. Whence the intersection of the given line and given plane will be in the intersection of the two planes. Finding then the projection 0*p* of this intersection (*Prop*. v.), we have the projections of two lines through the point required; and hence *p*, their intersection, is the projection sought.



The height (or figure) is known at once from the scale 0*p*, formed during the preceding construction.

Scholium. Problems may be proposed for solution in infinite variety; but the general principles of the method may be sufficiently understood from those which have been here given.

There is only one remark necessary: viz., that though for the sake of keeping up a view of the analogy between the Topographic Projection and the Descriptive Geometry, we have generally employed the *traces* of the line and plane as our points of departure; yet in practice it is quite sufficient that we take *corresponding figures* (as 5, 5; 8½, 8½; etc.), instead of the special ones 0, 0. The advantage gained by this consists in two things:—that it often saves the trouble of extending the scale back to 0 in the drawing; and that in consequence it requires less space and avoids damaging the paper. These details, however, belong more properly to the practical than to the theoretical branch of the subject, as do likewise the applications of the principles to actual contour operations, whether in respect to the surveys or the drawings.

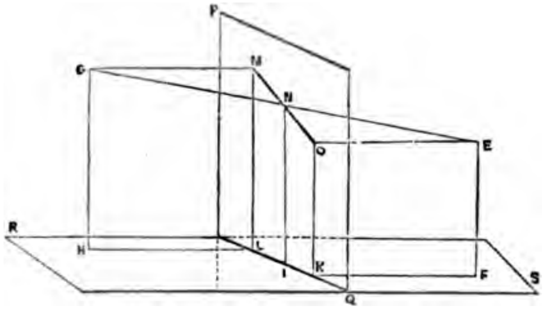
PERSPECTIVE.

SECTION I.

GENERAL PRINCIPLES.

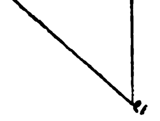
1. The business of Perspective, when divested of all irrelevant conditions, is, simply, to solve the following problem:—

Given the projections of a point on the horizontal and vertical planes, and likewise those of another point, on the opposite side of the vertical plane to find the intersection of the line joining them, upon the vertical plane.



Thus, let PQ be the picture-plane; E the place of the eye, given by the projectors EF, EO, or by FK, KO; G similarly the place of "the point to be put in perspective," given by the projectors GH, GM, or by ML, LH: then the problem is, to find on the plane PQ, considered as the plane of the drawing, the intersection of the line EG with the plane PQ.

2. The figure above given may be considered as a general picture of a model of the planes, points, and lines, drawn by parallel projectors upon a plane different from those upon which we are required to operate; in fact analogous to the figures employed in the pure theoretical geometry of three dimensions. Very simple considerations, however, enable us to make a figure more in accordance with the objects we have in view. Thus:—



Let LQ denote the same line as in the former figure, viz., the intersection of the picture plane with the ground or horizontal plane; denote by $e_1 e_2$ the point E, so that ee_1 = EO = FK, and ee_2 = EF = OK; and similarly G by $g_1 g_2$. We then have the problem converted into a very simple one of Descriptive Geometry, one of the planes of projection being that on which the perspective is required to be made.

The actual construction then becomes simply this:—

Join $e_1 g_1$ the horizontal projections of the points, meeting LQ in L; draw LN parallel to $e_1 ee_2$ meeting $e_2 g_2$ in N: then N is the perspective sought.

Viewed as a constructive process, this is direct and simple: for, either implicitly or explicitly (mostly the latter), the points g_1, g_2 are always given in every problem. No problem, indeed, is proposed in a complete state where this is not the case. When, however, points to be put in perspective are numerous, and when (as is mostly the case in practice) they have certain given relations to one another, some degree of modification will very much facilitate the *entire* process for a figure composed of many related parts. In fact, the determination on the horizontal and vertical planes, of the projections of those points when they are thus only implicitly given, takes away much from the simplicity of the whole work, notwithstanding the great simplicity of its final stages. It has, too, the practical disadvantage of requiring most of the operations to be performed on that region of the paper which is to receive the final picture—a disadvantage which a draughtsman of very moderate practice can fully estimate. This disadvantage is, however, common to most methods usually proposed, and is by no means peculiar to this; at the same time, as a key to the modified methods, this fundamental one must be thoroughly comprehended by the student, in order to which let him construct the following problems:—

(1.) Given the projections of a triangle to find its perspective, for three different positions of the eye, selected at pleasure.

(2.) Given the horizontal projection of a square, the vertical projection of one of the diagonals, and the place of the eye, to find its perspective.

(3.) Given a regular tetrahedron having one of its faces parallel to the picture, and the place of the eye, to construct it in perspective.

(4.) The eye and a cube are given, the cube having two of its diagonals perpendicular to the plane of projection; to put the cube in perspective.

(5.) One of the diagonals of a cube is perpendicular to the picture plane, and being produced, passes through the eye: find its perspective.

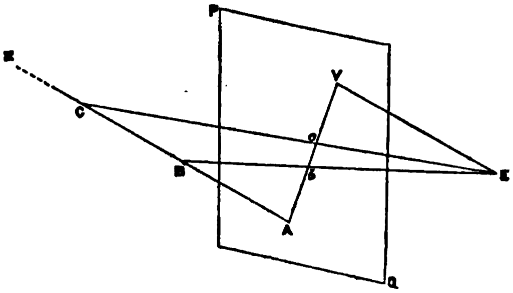
(6.) Show that for all positions of the eye in the prolongation of that diagonal, the perspectives of the cube will be similar and similarly situated figures.

3. In the method just discussed, we find the perspectives of isolated given points, and also those of rectilineal figures by drawing lines to join the perspectives of their angular points. In the next, this process is reversed: viz., into finding the perspective of a point considered as the intersection of the perspectives of two lines which intersect there; and the perspectives of a line as the intersection of the perspectives of two planes which intersect in that line.

As a first consequence of this system, we shall be required to find the perspective of a finite given line, not, as *in the first place*, limited by its given extremities, but as infinitely produced beyond the picture, and commencing at its intersection with the picture; and, similarly, the perspective of a plane is first found, supposing that plane to be infinitely produced, and commencing at its intersection with the picture. The mutual intersections of these perspectives, taken in proper order, give conjointly the perspective of the figure itself.

4. This mode of representation has already been referred to in the

pure Geometry, pp. 157, 158; and by recurring to what is there said, the circumstances just mentioned will become clear at once. Thus:—



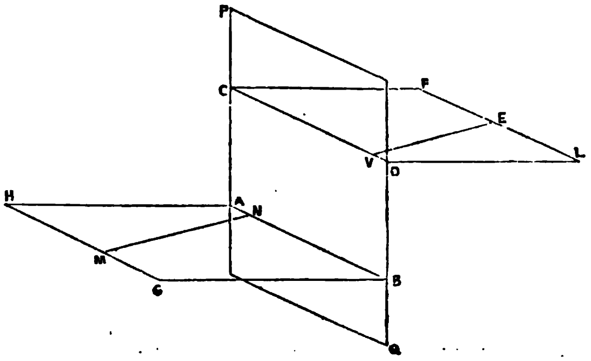
if E be the eye, AZ *any* straight line on the opposite side of the picture PQ: then the perspective of the infinitely produced line AZ from the picture at A will be one definite line AV, of which V is the termination.

For every line drawn from E to a point in AZ will lie in the plane drawn through AZ and E; and hence its trace on the picture will be a point in the trace of the plane EAZ upon the picture. That is, the perspective of every point in AZ is some point in the limited line AV; and the line AV will be the perspective of the entire infinite line AZ.

Moreover, as all the lines drawn from E to AZ must lie *between* the parallels AZ and EV, we see how V is to be determined; viz., by drawing EV parallel to AZ to meet the picture plane in V. This point, V, is called the *vanishing point* of the line AZ, from the perspective of AZ terminating or vanishing at that point.

It has been shown in the Geometry (*Pls.* I. 26) that all lines parallel to AZ have the same vanishing point V; and hence when the *inclination* (or direction) of a line is given to the picture and the place of of the eye also given, then one point in its perspective representation is given, whatever in other respects may be its situation. The trace of the particular line upon the picture plane gives another point A; and hence the entire perspective of the indefinitely extended line is given when its trace and vanishing point are given or found.

5. The case of the perspective of a plane is precisely similar in its general character.



Thus, let PQ be the picture, BH the plane to be put in perspective (supposed to be infinitely extended on the side of PQ opposite to the eye E) and AB its trace on the picture plane. Then if a plane CL be drawn through E parallel to HB, the trace of which on the picture is CD; the band of the picture plane, infinitely produced both ways, is the perspective of the infinitely extended plane of which HABG is a part.

For, as before, the perspective of every point in HABG is situated somewhere in that band; and there is no point in that band which is not the perspective of some point in HABG.

The line CD is called the *vanishing line* of the plane HABG, from the perspective of the plane terminating or vanishing in that line.

It follows, that all parallel planes have the same vanishing line.

Moreover, all lines situated in the same plane, as MN, *etc., will have their vanishing points, as* V, *etc., in the vanishing line of that plane.*

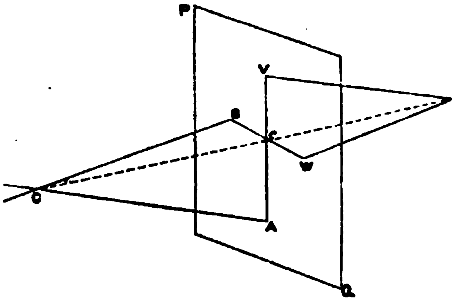
For HB, CL being parallel planes, and EV, MN parallel lines, one of which MN is (*Hypoth.*) in one of the planes HB, the other must be in the other (*Pls.* i. 4). Whence since EV is in the plane CL, its trace V must be in the trace CD of the plane CL. That is, the vanishing point of MN in the plane HB is situated in the vanishing line CD of the plane HB.

And again, all lines anyhow situated in any number of parallel planes, have their vanishing points in the common vanishing line of those planes.

These simple theorems will find important applications presently.

6. It only remains, in this part of the subject, to justify the remarks made at the opening of the section, by showing:—

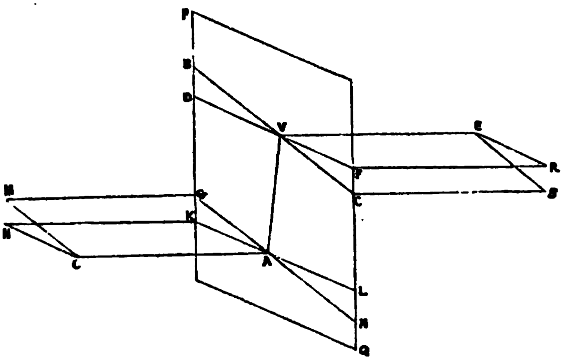
(1.) *The intersection of the perspectives of two lines is the perspective of the intersection of the two lines themselves.*



Let AC, BC be two lines which intersect in C, and have A, B for their traces on the picture plane; and let V, W be their vanishing points respectively. Then AV, BW will be their indefinite perspectives, intersecting in *c*; and it is alleged that *c* is the perspective of C, or, in geometrical language, that C, *c*, E are in one straight line.

This has been already proved at p. 157.

(2.) *The intersection of the perspectives of two planes is the perspective of the line in which those planes intersect.*



Let MA, NA be two planes which intersect in CA, and their traces on the picture-plane be GH, KL; and let VS, VR be planes through the eye E parallel to them, having EV for their intersection, and BC, DF for their traces on the picture-plane. Then the bands BCHG, DFLK are the perspectives of those planes, having the line AV for their intersection.

Now, since the planes MA, AN which intersect in AC are respectively parallel to the planes SV, VR which intersect in EV, the line EV is parallel to AC (*Pls.* I. 16). Whence V is the vanishing point of AC, and AV is the perspective of the line AC. Wherefore, the intersection of the perspectives of the two planes MA, AN is the perspective of their intersection AC, as stated in the enunciation.

We have next to show the method of defining the data of a perspective problem, considering all the lines employed to be traced upon the picture-plane of the paper. We cannot, indeed, exhibit the *immediate* data in the direct manner that we are able to do in the commencement of a problem in the Descriptive Geometry; but with very little subsidiary construction we are able to render the exhibition of a perspective problem in a light quite as simple and intelligible.*

7. In all direct problems in perspective, the position of the eye with respect to the picture forms part of the data. A perpendicular from the eye to the picture meets it in a specific point, called the *centre of the picture;* and a line through it parallel to the horizon (or the top and bottom of the picture itself) is called *the horizontal line.* When this point is not taken at, or very near to, the middle of the horizontal line, the effect of the picture is that of a painful distortion on account of the eye being

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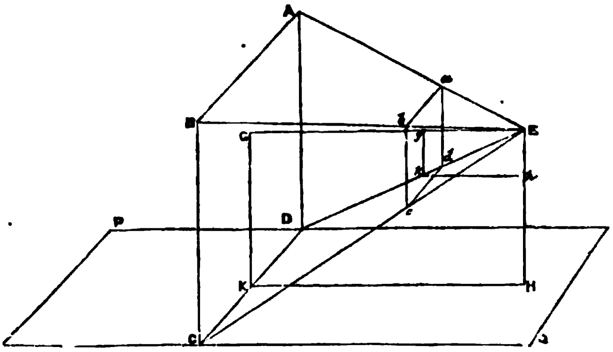
* In fact, all things considered, the Descriptive Geometry itself involves subsidiary considerations and constructions not less operose than those required by Perspective. Indeed, Descriptive Geometry is mainly conversant with inverse operations: operations upon the projections of the direct figures, so as to get the projections of the quæsita of the problem, and thence the quæsita themselves. English writers have confined themselves almost entirely to the projections of the figures which constitute the data of a problem: the French to the processes which are requisite for the determination of the quæsita. The French begin where the English writers leave off; the one is the complement of the other: neither is perfect and complete in itself; together they form a complete and perfect system. What the English "Orthographic Projection" is to the Descriptive Geometry, this determination of the working data from the absolute conditions of the problem is to the practical operations of perspective solution.

compelled to take in a wider range of scene on one side than the other. It is otherwise with respect to the height, inasmuch as we habitually refer all altitudes and depressions to our own horizon; and a scene which supposes the eye to be at the middle of the vertical height of the picture would be less satisfactory to the observer than one where it was far out of the middle of the horizontal breadth.

8. By the *distance of the picture* is meant the distance of the eye from it, or the length of the perpendicular drawn to it from the eye. This is sometimes marked on the horizontal line through the centre, each way from that centre; sometimes a circle with this distance as a radius is described on the picture-plane about that centre, and sometimes it is merely reserved without any separate exhibition.

Some degree of confusion is created by most writers neglecting to explain what is really meant by "the distance of the picture." It is not that 18 or 20 inches at which we hold the paper from the eye when making a sketch, or the distance at which we stand from the picture when viewing it from a proper position.

Let *abcd* represent the *frame* of the picture interiorly (or the visible boundary); E the eye; PQ the plane of the scene (ordinarily horizontal); EH the height of the eye above it; E*abcd* the pyramid of rays having the eye for vertex and the frame for base; CD the line in which *c*E*d* cuts the plane PQ; and ABCD the section of this pyramid by a plane through CD parallel to *abcd*. This plane ABCD is that upon which the perspective drawing is supposed to be made; whilst the picture (the actual paper or canvas) which is presented to our view is the "reduced picture," or picture of the real perspective, *abcd*.



Draw EG perpendicular to ABCD cutting *abcd* in *g*. Then *g*E is the distance at which we stand to view the actual picture; whilst EG is the real distance of the picture upon which the perspective is drawn. These may be called the *inspecting* and *constructive distances* respectively.

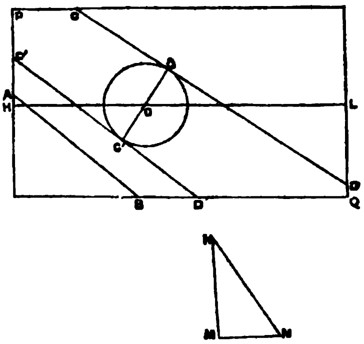
The pictures on ABCD and *abcd* are, however, similar, and all the parts of the one proportional to the corresponding parts of the other; as are, likewise, all lines connected perspectively with them. Draw the plane GEH cutting ABCD in GK, *abcd* in *gk*, and PQ in KH. Also draw *kh* parallel to KH. Then it follows from the similar figures, that if all the parts of the scene were diminished in the ratio of GE : *g*E, we might actually construct on the inspection plane *abcd*, In direct perspective, however, this is never done; but the drawing on a reduced scale is made upon *abcd* from that actually constructed on ABCD.

9. Perspective drawings are rarely made to any special scale (as an inch to a yard or to a furlong), and general landscape drawings never. The *practical* difficulty is not in these cases felt so strongly—often not felt at all. But the confusion of mind about the relation between ABCD and *abcd* is strongly felt by the reflecting student. Cases, however, do occur where it creates practical difficulty as well as theoretical.

In the inverse perspective it is found in a more troublesome form than in the direct; but of this hereafter.

10. A plane to be put in perspective is given by means of its trace on the picture-plane actually exhibited, and its inclination to that plane not exhibited but reserved. We are required, then, to connect the data with its perspective, thus giving rise to the following problem:—

Given the trace and inclination of a plane in respect to the picture-plane, together with the place of the eye: to find the vanishing line of that plane.



Let PQ be the picture; O its centre; HL the horizontal line (where LQ denotes the height of the eye above the horizontal plane in ordinary drawings); KM the distance of the picture; MKN the complement of the inclination of the given plane to the picture; and AB its trace upon the picture.

Draw MN perpendicular to MK; with centre O and radius MN describe a circle on the picture-plane; draw tangents to it, viz., CGD and C′G′D′, parallel to AB.

These are the vanishing lines of the plane according as the given inclination is towards O or the contrary, as is too obvious to need a formal statement of the proof here.

Cor. 1. If either of the tangents CD, C′D′ should coincide with AB, it indicates that the plane in question passes through the eye.

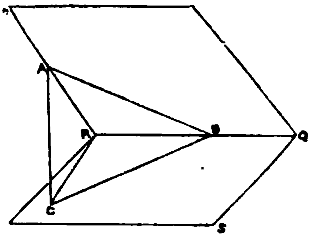
Cor. 2. When the plane to be put in perspective is parallel to the picture-plane, this mode of *complete representation* is inapplicable, inasmuch as such plane can have no trace (and consequently no vanishing line) situated on the picture. We can hence only exhibit the perspective of a limited portion of the plane; and this itself will require a modified process, which will be duly explained further on.

11. A *line* is most simply given for the purposes of perspective, when it is taken as *the intersection of two given planes.* Its entire perspective is then the intersection of the perspectives of the two planes: that is, its trace is the intersection of the traces of the planes, and its vanishing point the intersection of the vanishing lines of the planes, as has been shown.

12. A second manner in which a line is often given is, as *having a given position in a given plane.*

With a view to the analysis of the geometry of the problem, it will be advisable to view the planes in a general manner, apart from the perspective operations.

Let PQ be the given plane and RS the picture-plane; AB the given line in the plane PQ, and RQ the given trace of the plane, and B that of the line.



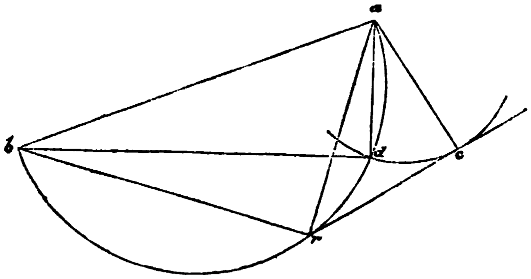
Take any point A in AB; draw AC perpendicular to the plane RS, and AR in the plane PQ perpendicular to RQ; and join RC, BC.

Now the plane ARC passing through the perpendicular AC is perpendicular to the plane RS; and hence ACR and ACB are right angles. Also, since AC is perpendicular to the plane RS, and AR perpendicular to a line RQ in it, the line CR is also perpendicular to RQ (*Pls.* II. 13). Wherefore ARC is the profile of the given dihedral angle of the object-plane and picture-plane, and is hence given.

Again, the point A in the given line being given (or assumed as given), together with the angle ARC, and ACR being a right angle, the lines AC, CR are given. The line AB being also given as well as AC, and the angle ACB being a right angle; the line BC and the angle ABC are also given. That is, the angle made by the given line with its projection, together with the projection itself, on the picture-plane are given.

This reduces the problem to another form which will be presently investigated; and hence we shall here only give the construction up to the stage already analysed. It is more conveniently performed as bye-work apart from the drawing.

In reference to the preceding figure, the small letters here will represent the capitals in the analysis.



Take *ab* equal to any segment of the given line, estimated from its trace *b*, as a working datum. Draw the perpendicular from *a* to the trace of the plane, and place *ar* equal to it in the semicircle on *ab*. Make the angle *arc* equal to the given profile angle of the plane and picture; draw *ac* perpendicular to *rc*, and set off in the semicircle *ad* equal to *ac*. Then *abd* is the inclination of the given line to its projection.

This construction is, obviously, only a performance upon one plane of operations indicated by the analysis as belonging to several; which process, where *magnitudes* only and not *positions* are concerned, is always to be adopted in practice.

13. A third method of defining a line for perspective purposes is :—

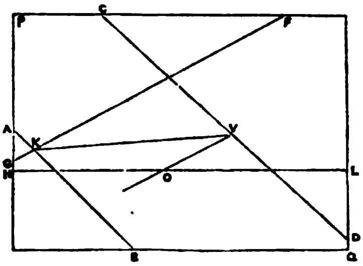
Given its plane, and its projection on the picture-plane, to find its perspective.

[Sometimes the angle which the line makes with its projection is substituted for the last datum.]

In this case we have virtually a second plane through the line given, as well as that which contains the line. This plane, moreover, being perpendicular to the picture-plane, its vanishing line passes through the centre of the picture, and is parallel to the trace, that is, to the given projection. It is therefore given in position. Whence the following construction :—

Let PQ be the picture-plane, AB the trace, and CD the vanishing line of the plane which contains the given line; let GF be the projection of the line on the plane of the picture intersecting AB in K; and let O be the centre of the picture.

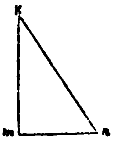
Draw OV parallel to FG meeting CD in V: then V is the vanishing point, and KV the perspective of the line.

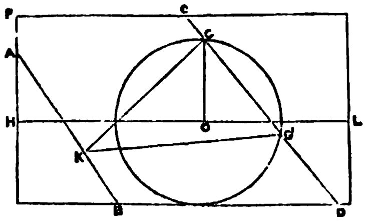


14. Finally, a line is given by *its containing plane, its trace, and its inclination to the picture-plane.*

A line drawn through the eye parallel to the given line will make an angle with the picture-plane equal to the given one; which angle will therefore be given. But the whole of the lines through the eye which do this, make equal angles with the perpendicular from the eye to the picture-plane; and hence they are situated on a right cone which has that perpendicular for its axis, and a circle on the picture-plane for its base. The centre of this circle is the centre of the picture; and the axis of the cone being given in magnitude, the radius of the circle is given in magnitude; and the circle itself is given in magnitude and position.

Whence the following construction :—





Let AB be the given trace of the containing plane; K the trace of the line; CD the vanishing line of the containing plane; and O the centre of the picture. Make *km* the distance of the eye from the picture; *mkn* the complement of the given inclination of the line to the picture; and *kmn* a right angle. With centre O and radius OG equal to *mn* describe a circle cutting CD in GG'.

Then KG or KG' will be the perspective of the entire line, according as the given inclination is towards the centre of the picture or in the contrary direction.

The unfinished part of the construction of the second case of the data is identical with this: for the inclination of a line to a plane is the same thing as its inclination to its own projection on that plane.

The perspective of a *point* is always viewed *practically* (that is, the operations for finding it always imply its being so) as the intersection of the entire perspectives of two lines which pass through it. This of course renders further remark unnecessary, as it leads simply to direct constructions of those two perspectives.

SECTION II.

PRACTICAL PERSPECTIVE.

In general the application of perspective is to objects having certain features of regularity and symmetry which enable us greatly to abridge the actual work that would be required to put each angular point (as an isolated point) in perspective. For instance, nearly all the planes in which lines or points are actually given are either parallel or perpendicular to the horizon; or their relations to such planes are easily determined both in fact and in perspective. Again, most of the plane figures to be put in perspective are rectangles, or have easily assignable relations to rectangles or to parallelograms; and most of the solid angles that occur are trihedral right angles, having one of the faces horizontal. Even where these conditions are not fulfilled, the data can almost invariably be connected by subsidiary processes with data that do fulfil them; and in any particular case where even this may fail, the general methods already given will be adequate to a full and perfect construction of the perspective.

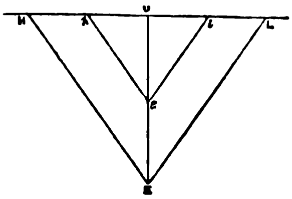
In the case of curves, as it is rarely that any one but the circle occurs in the practice of the draughtsman or engineer, it may be sufficient to state that it is deemed in practice sufficient to find the perspectives of eight points equidistant from each other (the angular points of an inscribed regular octagon), and then carefully tracing by hand an ellipse through them.

A few incidental notes, which though not geometrical perhaps as to strict form, are yet founded on geometrical and physical principles, may be given *in initio*. For the sake of reference, they are designated as Propositions.

PROPOSITION I.

To prepare the drawing and its scales.

(*a.*) To eyes of ordinary conformation, horizontal vision is painful and indistinct when the bounding line of light makes an angle of more than 30° with the optic axis. When, therefore, the breadth of the picture is fixed upon, the distance of the eye from it, to form an agreeable perspective, becomes also determined—at least within very narrow limits of variation; and when the distance of the eye is first fixed upon, the breadth of the picture becomes, conversely, dependent. If this rule be rigidly adhered to, the relation will be exhibited by the following construction; and if varied, the effect of the variation can be readily traced.



If the distance OE be given, and HL be a horizontal line through the centre of the picture; take any equal distances O*h*, O*l*, on each side of O, and describe the equilateral triangle *ehl*; draw O*e*, and make OE in it equal to the given distance of the picture; and draw EH, EL parallel to *eh*, *el*. Then H, L are the lateral boundaries of the picture.*

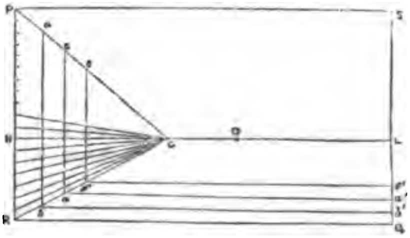
(*b.*) The *height of the eye* above the horizon is usually between five and six feet, and when the supposed observer stands on the same horizontal plane with the building to be put in perspective, this height becomes a fixed standard for forming all the scales of the picture. This in fact it is which forms the fundamental scale; the distance of the eye, and consequently the breadth of the picture, being regulated by it.

The height of the top of the picture above the horizontal line is a question less easy of decision. Of course it may include an angle of 30° on the ground of our distinct vision being limited by a right cone of rays, the axis of which cone is the perpendicular from the eye to the picture. But in fact, from our being so habituated to elevate the optic axis itself that we do it unconsciously to some extent, we can take in without pain a more extended space vertically than laterally.

(*c.*) Having fixed upon the boundaries and marked the centre of the picture, draw a line through that centre horizontally. This is called the *horizontal line.*

(*d.*) *To form a scale of heights.*

Let HR be taken of the height of the eye, and divide the entire line PR in conformity with it; draw lines to any point G in HQL: then the lines *ab*, *cd*, *ef*, etc., are the perspective heights of the line PR when removed backwards from the picture, so that the perspective of R shall be distant from the picture so far as to be represented on the horizontal lines *bb'*, *cc'*, *dd'*, etc.



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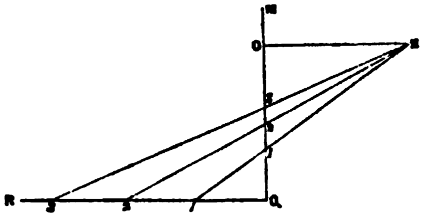
* If the horizontal breadth be taken much more than this, it cannot be seen in all its parts without moving the axis of the eye so as to see one part separately from another (which would be tantamount to looking at the perspective of a perspective), or else distressing the eye to attempt to take it all in at one view. If, on the contrary, it be much less, the picture has precisely the same effect as an agreeable picture would be made to have by cutting off a portion at each end—a mutilated appearance.

(*e.*) *To form a scale of distances.*

Let the figure be supposed to be a section of the entire system by a plane perpendicular to the ground plane, passing through the eye. It may be considered, then, as a *subsidiary figure* in a plane at right angles to the picture and to the horizon.

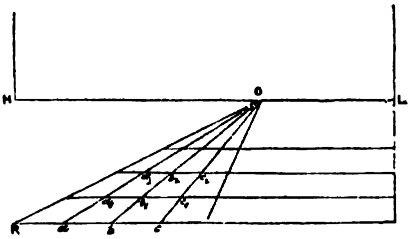


Set off upon the line RQ in which this plane intersects the horizontal plane the several distances Q1, Q2, Q3, etc. Make QO (from the same scale) equal to the height of the eye; and OE parallel to QR equal to the distance of the picture. Then E1, E2, E3, etc., will cut from QO segments Q1, Q2, Q3, etc., which represent on the picture the corresponding distances on the horizontal plane.

The proof is obvious.

(*f.*) *To form a scale of breadths.*

Divide the ground line RQ per scale; draw lines RO, *a*O, *b*O, *c*O, etc., to the centre O (or any point in HL): then if 1, 2, 3, etc., be the divisions for distances, and parallels be drawn to RQ, as in the figure, $a_1 b_1$, or $b_1 c_1$, etc., will be the perspective breadth at the distance 1. Similarly $a_2 b_2$, or $b_2 c_2$, etc., will be that at the distance 2; and so on.



These scales, though often used by practical draughtsmen, are not essential. Though given here in accordance with such frequent usage, they are not recommended to the draughtsman, who is also a geometer. When used, it is better to keep them as subsidiary figures than to encumber the actual drawing with them.

PROPOSITION II.

To put a plane figure in perspective.

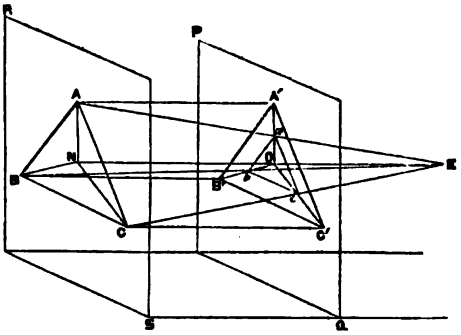
As a general statement of the rule, it may be directed to put each *entire* component line of a rectilnear figure in perspective. These perspectives, by their mutual intersections, give the perspective required.

In the case of curves, set a series of points sufficiently near for the purpose, and having put these in perspective, trace a curve through them by hand or by instruments, as the case may admit.

In practical perspective, however, we have seldom to operate upon figures which do not possess some kind of symmetry, which greatly simplifies the entire work by diminishing the number of independent steps. Thus the ground-plan of a building is mostlycomposed of rectangles, or of figures somehow related to rectangles; the walls are invariably rectangular, and the tops and bottoms of a range of windows are each in line; the roofs are rectangles, or occasionally isosceles triangles; and so on. It is the same in machinery, in a great degree, for even wheels do not often occur except in horizontal or vertical positions. On this account directions somewhat more special may be given for operating.

(*a.*) *Let the plane of the figure be parallel to the picture.*

This, on the general principles of perspective, is an isolated or exceptive case; since the vanishing line of the plane itself being infinitely distant, cannot be brought into use. It is usually waved in practice by making it dependent on the construction of another case; viz., by finding the perspectives of some of the points by feigning them to be the intersections of the trace of the plane on the ground-line and some other lines. The following construction is more simple in principle as well as in practice.



Analysis.—Let PQ be the picture-plane, RS a plane parallel to it, containing the figure ABC to be put in perspective: draw lines, AE, BE, CE to the eye E, meeting the picture in *abc*. Then the figure *abc* will be the perspective of ABC, by the definition.

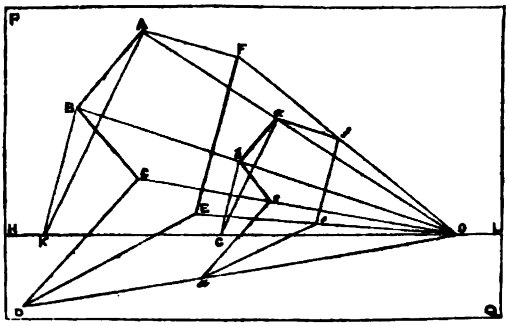
Draw perpendiculars AA′, BB′, CC′ to the picture, and likewise the perpendicular EON to the picture, meeting the picture at O, and the parallel plane at N. Draw also the planes ANE, BNE, CNE, cutting PQ in A′O, B′O, C′O.

Then the lines AN, A′O, AA′, NOE, A*a*E are in one plane; and hence since this plane cuts the parallel planes, each line in one is equal and parallel to the corresponding line in the other, namely, A′B′ to AB, A′O to AN, etc. Also since AE, A′O are in one plane, the point *a*, which is the perspective of A, is in the line A′O; and similarly *b*, *c*, etc, are in B′O, C′O, etc.

Again, since *a*O is parallel to AN, the triangles EO*a*, ENA are similar, and hence

AN : *a*O : : EN : EO, a given ratio.

Now O is the projection of the eye upon the picture and A′B′C′ is the projection on the picture of the figure to be put in perspective. Wherefore, if we take O the centre of the picture, and describe the given figure on the picture-plane in its appropriate position, we shall get the following construction :—

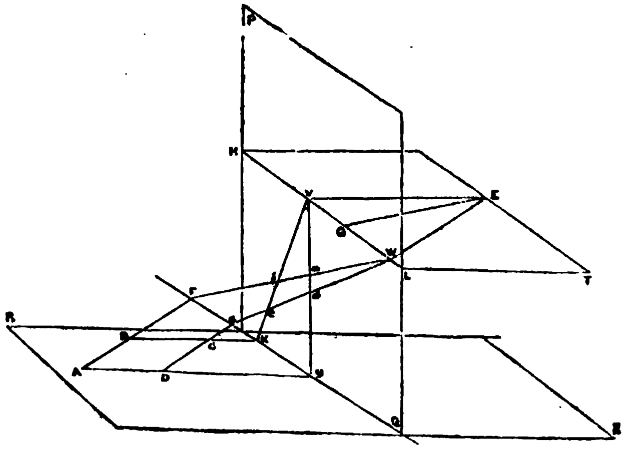


Let ABCDEF be any figure, regular or irregular, to be put in perspective, and O the centre of the picture, PQ. Draw AO, BO, . . FO ; make OG (in any direction, but generally in the horizontal line as most convenient) equal to the distance of the picture, and OK that of the plane of the given figure from the eye. Draw KA, and G*a* parallel to it meeting AO in *a*: then *a* is the perspective of A.

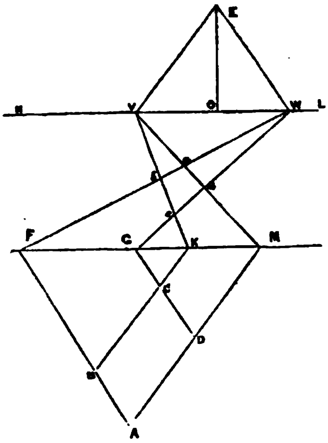
The concluding part may either be effected similarly (as is done in the figure for *b*) : or still more simply by drawing *ab* parallel to AB, *bc* to BC, etc.

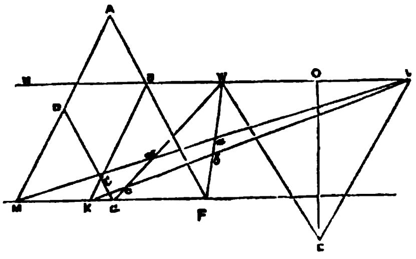
For the figures ABCDEF and *abcdef* are evidently similar, and have their homologous sides parallel. Also their sides are to one another as OG to OK, that is, as EO to ON of the former figure. Wherefore all the conditions of the entire figure, being the perspective of the former, are fulfilled.

When the given figure is in any respect regular, the process is or-



Let ABCD be, for instance, a parallelogram; PQ the picture-plane, and the other parts as before. Draw EV parallel to BC or AD, and EW to AB or DC. Then, completing the figure, *abcd* is the perspective of ABCD.





Let, now, the planes HT, RQ simultaneously revolve about HL, KQ, respectively, so as always to be parallel to one another. If the direction of their revolution be such as to bring them finally into the *extension* of PQ, the parts bearing the several letters quoted in these figures will take the positions in Fig. 1. If, on the contrary, they revolve in the opposite directions, they will ultimately coincide with the picture plane as in Fig. 2.

Now the line FG or trace of the ground-plane, the horizontal line HL and centre O of the picture, and the distance EO of the eye from the picture are all given. The figure ABCD is also given, and hence the traces F, M, K, G, of its sides are also given. But EV is drawn through a given point E parallel to a given line AD or BC, and meets the given line HL in V. Whence V is a given point. Similarly W is a given point. Whence the *entire* perspectives of the sides of the figure ABCD are given; viz., KV, MV, those of BC, AD, and FW, GW those of AB, CD. Wherefore, their intersection gives the perspective sought. Whence, the rule.

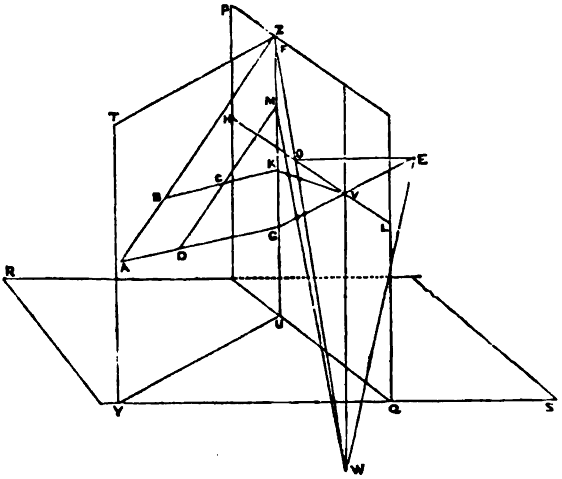
Place the given figure in its appropriate position on the ground-plane; from O the centre of the picture draw OE at right angles to the horizontal line and equal to the given distance of the picture. Produce the sides of the figure to meet the trace of the ground-line, and parallel to them draw lines to meet the horizontal line. They will intersect it in the vanishing points of those lines. Having now the traces and vanishing points of the lines, draw their entire perspectives; and these, by their intersections, give the perspective sought.

The two figures above give the same result. The first interferes less with the part on which the drawing is to be made, as well as affords more distinct work: but it on the other hand requires more space above and below the picture for tracing the component parts. When possible to be used, it is always the preferable.

The process is the same too for *any* horizontal plane. The only difference in any respect is, that the trace of the plane itself will be parallel to the ground-line instead of coincident with it.

SCHOLIUM. Moreover, there is nothing in the *reasoning* upon which this process is founded to confine the plane which contains the figure to be a horizontal one, or the picture to be perpendicular to the horizon. It is hence universally applicable, whatever be the inclination of the object-plane to the picture-plane.

(*c*) *Let the plane of the figure be vertical*, but not parallel to the picture.



Let the ground-plane, etc., be as before, as far as the parts are common; and let the traces of the vertical plane TU upon the ground and picture-planes be YU and UZ. Draw EV parallel to UY; and then VW parallel to UZ. This will be the vanishing line of the plane TU.

Let ABCD be any figure (a parallelogram is taken for example) in the plane TU. Find its perspective precisely as in the last case; for which purpose the lines concerned are given, the student to fill up the description.

The part in the actual construction in the drawing, too, is so completely a repetition of the last case, that it is left for the student to execute in detail.

(*d*) For sloping planes, such as roofs, etc., the process is exactly the same.

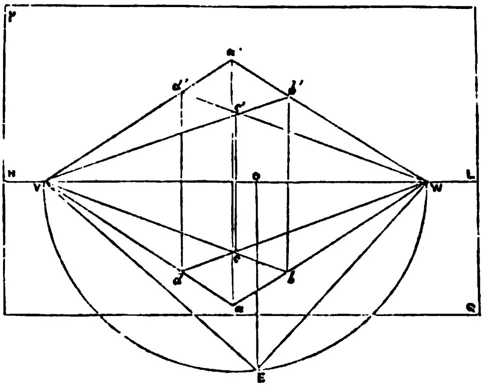
SCHOLIUM. It has been no part of the plan of this section to furnish examples; or even to intimate what simplifications can be made under any special circumstances of a piece of architecture or of a system of machinery. These are often numerous and beautiful; but they will be furnished by the drawing-master, whilst our business is confined to the subject simply as one of pure geometry.

MISCELLANEOUS PROPOSITIONS.

PROPOSITION I.

Assuming that the centre of the picture is in the middle of the horizontal line, to find the distance of the eye, there being some one regular building represented in the drawing.

Let *abcd* be the base, and *a'b'c'd'* any horizontal course of the building (as the eaves, the tops of windows, etc.); produce *ad*, *ab* and *a'd'*, *a'b'* to meet in V and W. Then these are the vanishing points situated in the horizontal line. The horizontal line hence becomes known, and consequently its middle point O.



On VW describe a semicircle, and draw OE perpendicular to VW meeting it in E. Then EO is the distance of the eye for which the perspective was formed.

This will be seen at once, if we suppose the semicircle and its lines to be revolved into a position at right angles to the picture.

Scholium. If there be two independent buildings in the picture, it may be tested whether the centre be taken truly in the middle of HL.

For we have only to perform a similar operation upon the second figure: and the intersection of the semicircles will give E, and hence also O.

It will be a good exercise for the student to determine the *effects* of taking a wrong point from which to view the picture. For instance:—

(1.) If the eye be taken so as still to retain the same things, except the point O, then the figures *abcd*, *a'b'c'd'*, etc., will still be the correct representations of parallelograms in the same planes; but they cease to be rectangular ones, and no two positions of the eye would cause them to represent the same parallelogram.

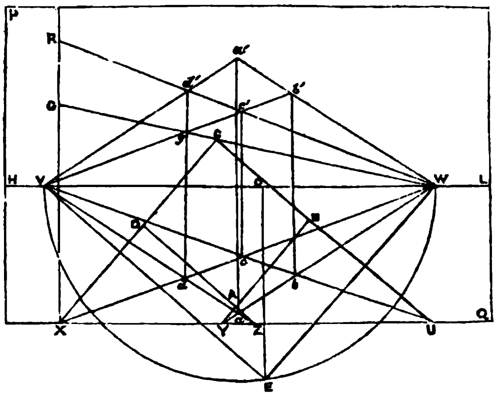
(2.) The same indeed is *generally* true of positions of the eye anywhere in the plane through E and HL: the exception being a certain circle in that plane, which it is left for the student to find; and even this is an exception which can be hardly considered absolute.

(3.) If the eye be taken without the plane of E and HL, the perspective can in no case suggest *abcd* to be a parallelogram.

PROPOSITION II.

To restore the original figure in its proper position.

Produce the sides of *abcd* to meet the bottom of the picture, viz.



as in the figure in X, Y, Z, U. Through X, Y draw lines parallel to EW; and through Z, U lines parallel to EV. These by their intersection give the figure ABCD, which is the original of *abcd*.

[This, it is obvious, is but a reversal of the process of finding the perspective *abcd* of the original ground plane ABCD.]

Again, if *a'b'c'd'* be the upper surface of the prism on ABCD (original), produce W*d'* to meet a perpendicular at X in R. Then XR is the height of the original prism, to the same scale as before.

Also, the height upon XR, which is represented by any part *dg* of the perspective *dd'*, will be denoted on the same scale by XG. And similarly with respect to lines lying in any other direction as well as the vertical.

PROPOSITION III.

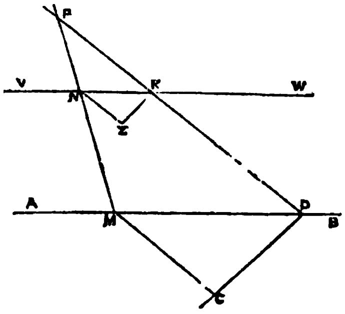
Two lines are given in perspective, to find whether they are in one plane.

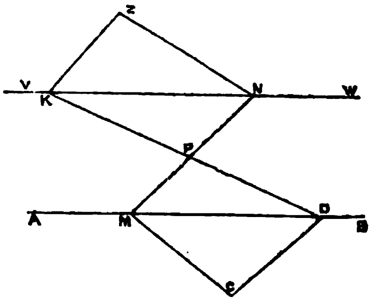
Join their traces, and likewise their vanishing points. If the two lines so drawn be parallel, the original lines are in one plane: but if not, they are not.

PROPOSITION IV.

Given the perspective of a plane and likewise that of a line, to find that of the point in which they intersect.

Let ABVW be the perspective of the plane, and CZ of the line. Now every plane drawn through CZ to meet ABVW will pass through the point of intersection. Let then CDZK and CMZN be *any* two such planes put in perspective; then the point sought is in the intersection DK and MN of these planes with ABVW; and hence at P their intersection.

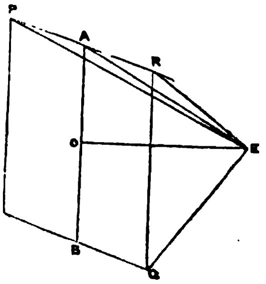


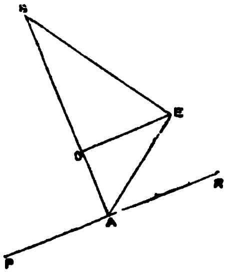


PROPOSITION V.

To assign the relations that must subsist between the perspectives of a line and plane, that the originals may be perpendicular to each other.

All lines perpendicular to a plane are parallel to one another, and hence have the same vanishing point; that is, the vanishing point of any one of them is the vanishing point of all. In the same manner the vanishing line of any one of a set of parallel planes is the vanishing line of all. It is hence immaterial which of the lines and which of the planes, having each its own proper direction, we take for the determination of the vanishing point and line; and hence we are at liberty to take those which offer the greatest facilities for construction. These will readily appear to be the line and plane which, when produced, pass through the eye itself: the trace of each then coinciding with the vanishing point and vanishing line respectively. The relation between these (the vanishing point and vanishing line) will hence be that which expresses the condition sought.





In the left-hand figure suppose PR the vanishing line of the plane, E the eye, and OE the perpendicular from E to the plane. Draw OA perpendicular to PR; join EA; produce AO; and in the plane EOA draw EB perpendicular to EA, meeting OA in B.

Then, since EO is perpendicular to PQ, and OA perpendicular to a line PR in it, the line AE is perpendicular to PR; and hence, the plane AOE is perpendicular to PR; and hence, again, to every plane passing through PR (*Pls.* II. 13, 16), as to EPR. The line EB in the plane AOE perpendicular to AE in the plane PER is that which gives the vanishing point of the lines perpendicular to PER. Wherefore we have the following construction:—

In the right-hand figure, let PR be the vanishing line of the plane, O the centre of the picture, and B the vanishing point of the line perpendicular to the plane. Join BO and produce it to meet PR in A; draw OE perpendicular to BA, and make it equal to the distance of the picture; and join AE, BE.

Now if BA be perpendicular to PR and at the same time BE to AE, the given line and plane are at right angles: if not, not.

If BA be perpendicular to PR, but BEA not a right angle, still a position (the centre of the picture remaining the same) can be found for the eye, which will render PR and B the vanishing line and point of a plane and line perpendicular to each other.

But if BA be not perpendicular to PR, then the condition cannot be fulfilled without changing both the centre of the picture and place of the eye.

Scholium 1. It is unnecessary to give the details of the solution of the problem:—*Given the eye, and either the vanishing point or vanishing line to find the other.*

Scholium 2. It is obvious that the traces are the only things left arbitrary with respect to the line or to the plane concerned; and that these, when once fixed upon, identify the particular line and plane under specific consideration. The trace of the line may be *anywhere* in the picture; and likewise that of the plane, subjected only to the condition of being parallel to the given vanishing line.

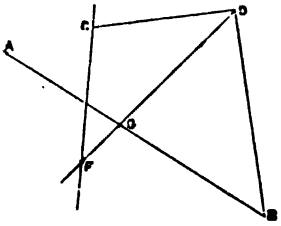
PROPOSITION VI.

Through a given point (in a given line) to draw a parallel to another given line.[*]

Let A be the trace, and B the vanishing point of the given line in which the given point whose perspective is G is situated; it is required to draw a line through G parallel to the given line whose trace and vanishing point are C and D respectively.

Join DG, DB, and through C draw CF parallel to DB to meet DG in F. Then F is the trace, and D the vanishing point of the line sought.

For, since the lines CD and GD have the same vanishing point D, they therefore represent parallel lines.



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[*] The reader is requested to give attention to the condition included in the parenthesis. A point is not given by its perspective on the picture-plane alone: for that perspective may represent *any point whatever* in the line which passes through that point and the eye. Another condition is therefore necessary for fixing the point as to actual position; and the most usual, as well as the most convenient one is, that it shall be situated in a given line; that is, in a line whose trace and vanishing point are either given or can be found from the data.

The same holds good with respect to any segment of the perspective of a line. That perspective may represent innumerable different lines; and it becomes in the same manner necessary that it shall be given in a given plane; that is, in one whose trace and vanishing line are either given or can be found.

Again, since AB, GD represent intersecting lines, the originals are in one plane; and the line DB through their vanishing points is the vanishing line of that plane.

The point C being in the trace of that plane, and CF drawn parallel to the vanishing line, it is the trace of the plane itself.

Wherefore F is the trace of the line GD in that plane.

PROPOSITION VII.

Through a given point to draw a line perpendicular to a given plane, and to find the point of intersection.

(1.) Find the vanishing point of the perpendiculars (*Prop.* v.).

(2.) Find the trace of the perpendicular through the given point (*Prop.* vi.).

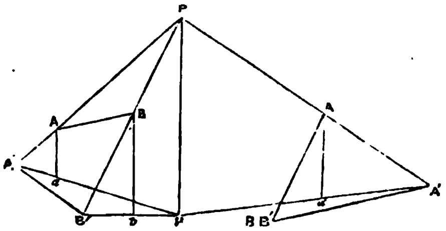
(3.) Find the intersection of the perpendicular and plane (*Prop.* iv.).

The construction is then complete, and composed of steps which have been already demonstrated.

PROPOSITION VIII.

To find the shadow of a given line on a given plane, the light emanating from a given point.

Let P be the luminous point, and AB the given line. Draw perpendiculars Pp, Aa, Bb to the given plane (*Prop.* vii.). Draw PB, pb to meet in B′, which will be the shadow of B; and PA, pa to meet in A′, which will be the shadow of A. Join A′B′, which will be the shadow required.



This construction is adapted to all cases; but it becomes something less complex as to operation when the intersection of the given line with the given plane is also given. For then B and B′ coalesce, as in the right-hand figure.

In this way the shadows of the angular points of any polygon or polyhedron may be found. When the line or surface is curved, recourse is generally had to finding the shadows of prominent points of the figure, and tracing a curve through them by hand.

PROPOSITION IX.

To find the shadow cast by a given line on a given plane, the rays (like sun-light) being taken as parallel.

This problem differs but little from the preceding in its general principle of construction, though in some respects the operations are rather more complicated, whilst in others they are somewhat simpler. The former construction was decribed in detail; but it will be easy for the student now to follow out the details from a general description of the operations to be performed. At the same time it will be desirable to notice particularly one or two special cases.

The problem divested of technicalities becomes the successive applications to different points of the figure of the following operation:—

Given a point, to find the intersection with a given plane of a line drawn through that point, parallel to a given line.

This problem requires the vanishing point of the pattern-ray to be substituted for that of the perpendiculars in *Prop.* V. It then becomes precisely identical with *Prop.* VI., as far as determining the trace of the line so drawn is concerned. Having, then, this line, its intersection with the given plane is found as in *Prop.* IV. The point thus found is the shadow sought.

The question as to whether the light comes in a direction on the same or a different side of the picture-plane only varies the question so far as the perspective of the pattern-line is concerned: though writers often make two separate cases of it, which is perfectly unnecessary. There is, however, one case, namely, that when the pattern-ray is parallel to the picture-plane, that requires a separate consideration; as the trace and vanishing points have here no existence at a finite (and therefore available) distance. With that case we shall close the subject.

Now all lines (as these rays) parallel to the picture and to one another are represented on the picture by parallel lines. The point through which any one of these passes being therefore given, the direction of its representation on the picture is also given. The process, therefore, differs from the general case already described in this particular only: that the rays, instead of emanating on the picture from the vanishing point, are all drawn parallel to one another.

SCHOLIUM. The inverse class of perspective problems has attracted little attention from writers on perspective in this country. Still when even *practically* viewed, this is quite as important as the direct one. The two classes stand in a relation to each other analogous to that of the direct orthographic projection, and the operations known as Descriptive Geometry; and for completeness of system both classes of the problems are as indispensable as in the orthographic. The student who wishes to pursue the subject further, is referred to *Cousinery*, *Géométrie Perspective*, 1828; and to a work under the same title by *Dufour*, of Geneva, 1827.

CONCLUSION.

It is not irrelevant to caution the student against indulging in a course of speculative discussions that seldom fail to be brought before his attention, respecting the "fallacies" or "illusions" of perspective. These are, one and all, only so many mental delusions, arising out of a confused or erroneous view of the character of perspective, and the geometrical properties involved in the formation of a picture. Most books contain some or other of these; often, too, in a very plausible form: and they are in each age revived as so many new discoveries, to be again refuted as they had been twenty times before.

These chiefly turn upon the following mistakes:—

(1.) Confounding the picture on the retina with that on the canvas. This gives rise to the sophisms respecting straight lines being represented on the picture by curves, and many others.

(2.) The supposition that we have the power in the same picture to move the axis of the eye to the right or left. This is a fertile source of fallacy, and contrary to the fundamental principle of perspective, that a picture is only composed of such scenery as can be taken in with the eye in one fixed position. This supposition is the foundation of the *panorama*, but not consistent with the idea of a *picture.*

(3.) Speculations respecting the diminution of the optic angle often lead to very absurd fallacies, especially where the plane of the objects is parallel to the picture-plane. If the student should encounter the oft-quoted example of a row of equal columns parallel to the picture, or two flag-staves stuck upright in the ground: then let him remember that the optic angle is diminished in the picture and in the objects precisely alike; and that in order to this being the case, the most distant pillar must have the same linear diameter on the picture that the nearest has, and that the flag-staves must be equidistant at top and bottom on the picture as well as in the actual position which they occupy. The whole confusion arises from an imperfect conception of the problem, and of the distinction always to be made between linear and angular magnitude, as things *per se* different from each other.

(4.) Much confusion has arisen, too, from overlooking the fact that whether it be a scene or the picture of that scene that is before us, we actually see with the same instrument, the eye, in both cases. If we then suppose, or even admit, that the eye does in any specific way modify the sensible form of the scene, it must also modify the picture (or perspective) in precisely the same manner.

Other sources of fallacy also present themselves occasionally; but the cautions above given will be sufficient for most cases, and moderate care will suffice for all others.

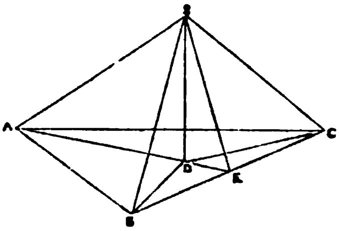
ISOMETRIC PROJECTION.

"A bird's-eye view of a scene" is a common and well understood expression for a view obtained from a point considerably elevated above the plane of the scene itself, so as to include also the interior (as well as the exterior) of the works which were concealed from the eye at a less elevation. Such a view, however, if represented on a plane, would be a perspective view; and though it would give a good general notion of the interior structure, would be too much distorted to be of use in reference to construction, where any close degree of accuracy was required. Like perspective under ordinary circumstances, it has suggested the employment of orthographic projection as a substitute; for the isometric projection is, in reality, only a special case of the orthographic, the plane of projection itself being restricted to one single position.

If the dihedral angles of a given trihedral right angle be bisected, the bisecting planes will pass through the same line. A plane perpendicular to this line is taken as the plane of projection, the projectors being parallel to the line itself. If the three faces of the trihedral angle be the ground and the two walls which form the angle of the building, the line in question will rise obliquely above the ground; and an eye placed anywhere in it to view the interior of the walls and ground may be said to have a bird's-eye view of the structure. Any point in either of the planes projected on the forenamed plane by lines parallel to the line instead of by lines radiating from the eye, will change the perspective into an isometric projection of the point.

The *axis of projection* is the line in which the three bisecting planes intersect; and it is easy to prove that it is equally inclined to each of the edges of the trihedral angle; and likewise equally inclined to each of the faces. We have now to ascertain what each of these inclinations is.

Let the right angles ASB, BSC, CSA of the trihedral angle be denoted as in the figure (they are seen obliquely, as they could not be otherwise exhibited without a model); make AS, BS, CS all equal, and denote their common length by a; draw the plane ABC, and the perpendicular AD to it from S; and draw the plane ASD, cutting ABC in AE, and BSC in SE.



Then ABC is evidently an equilateral triangle, whose centre is D, and whose side BC is bisected in E. Moreover the plane ABC is parallel to the plane of projection, and SD is the axis of projection.

Then, $\mathrm{AB} = \mathrm{BC} = \mathrm{CA} = a\sqrt{2}$;

$$\mathrm{AD} = \frac{\mathrm{BC}}{\sqrt{3}} = \frac{a}{3}\sqrt{6};$$

$$\mathrm{DE} = \tfrac{1}{2}\,\mathrm{AD} = \frac{a}{6}\sqrt{6};$$

$$\mathrm{DS} = \sqrt{\mathrm{SA}^2 - \mathrm{AD}^2} = \frac{a}{\sqrt{3}};$$

$$\mathrm{ES} = \sqrt{\mathrm{SD}^2 + \mathrm{DE}^2} = \tfrac{1}{2}a\sqrt{2}.$$

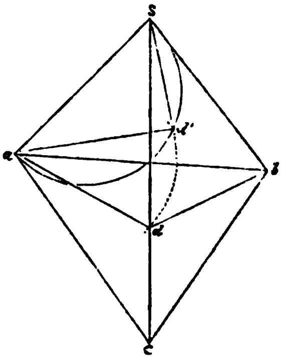
These elements serve for the determination of everything relating to the subject. Thus, for instance, we have

$$\sin \mathrm{SED} = \frac{\mathrm{SD}}{\mathrm{SE}} = \frac{\dfrac{a}{\sqrt{3}}}{\dfrac{a}{\sqrt{2}}} = \tfrac{1}{3}\sqrt{6} = \cdot 8164966 = \sin 54^\circ\ 44'.$$

$$\sin \mathrm{SAD} = \frac{\mathrm{SD}}{\mathrm{SA}} = \frac{\dfrac{a}{\sqrt{3}}}{a} = \tfrac{1}{3}\sqrt{3} = \cdot 5773503 = \sin 35^\circ\ 16'.$$

Which are the inclinations of an edge and a face respectively to the plane of projection; and agreeing with the fact that the triangle ASE is right angled at S, so as to render it necessary only to calculate one of these inclinations from a formula.

To *construct the isometric angle* is, however, more compatible with the geometrical character of the operations than to *compute* it; the more especially as we at the same time obtain the *lengths* of all the lines concerned in reference to any assumed magnitude of the edge SA (or SB, or SC). This is effected very simply in the following manner.



In *as*, *sb* taken at right angles, make *as*, *sb* equal to AS, SB of the former figure; join *ab*, and describe the equilateral triangle *abc* upon it. This is equal to the triangle ABC.

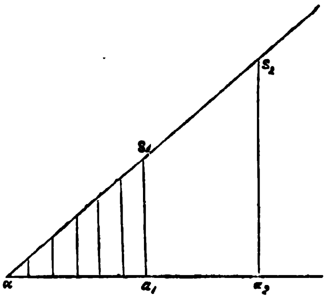
Find the centre *d* of the triangle *abc*, and join *ad*, *bd*, *cd*. Then these are equal to AD, BD, CD.

On *as* describe a semicircle, and in it set off *ad'* equal to *ad*, and join *sd'*. Then, *sad'* is equal to SAD, *asd'* to SED, and *sd'* to SD.

For, since *sd'a* is in a semicircle, it is a right angle; and we have $sa = \mathrm{SA}$, and $ad' = ad = \mathrm{AD}$: wherefore the two right-angled triangles SAD, *sad'* have two sides of the one equal to two sides of the other, and hence the third sides SD, *sd'* are also equal, and the angles opposite to these equal, viz., *sad'* to SAD.

Also SED being the complement of SAD, and *asd'* of *sad'*, these are also equal.

We are now able to construct the *isometric scale.*



Make the angle s_1aa_1 equal to the isometric angle by the preceding; on s_1a set off the units of the natural scale (suppose inches); and draw s_1a_1, s_2a_2, etc., perpendicular to aa_1. When aa_1, a_1a_2, etc. will be *isometric inches*:—that is, on an isometric drawing those lines will represent real inches in the model or other machinery delineated. They will also represent such parts of a building or range of works as are delineated to the scale of one inch.

If there be frequent repetitions of this class of operations, this scale should be carefully made of wood or brass, with a sliding bar perpendicular to aa_1, perhaps in the manner of the T-square. This being so moved as to pass over any division on as_1, as s_2 it would mark the distance aa_2 to be taken as its correspondent isometric distance.

For subdivisions, as eighths or tenths, a smaller bar, limited to pass over only as far as as_1 might be also used. Both bars may be fixed so as to move in grooves on the triangular plate which forms the scale.

The properties of this projection which (as far as is yet known) can be brought into practical use are few and simple; and it would be out of place here to give those which are merely curious. To obtain those few let us recur to the first figure, page 380.

1. The three equal edges SA, SB, SC, being equally inclined to the plane of projection, are projected into equal lines AD, BD, CD;

or they are isometric. On this account the three lines AD, BD, CD have been called *isometric axes.*

2. The three right angles ASB, BSC, CSA are projected into three equal angles ADB, BDC, CDA. Each of them (as they together form the angular space round the point D) is one-third of four right angles.

3. If any parallelogram be formed, two of whose sides shall be coincident with two of these lines AD, BD, CD, the other pair of opposite angles will each be one-third of two right angles.

4. The projection of any line parallel to any one of the rectangular axes SA, SB, SC, will be isometric with the axes of the projection; and no line which is not parallel to one of these can be isometric with them. (These several lines being supposed situated on the one or other of the faces of the trihedral angle).

For parallel lines projected on the same plane are proportional to their projections.

And as lines not parallel to these are not inclined to the plane of projection in the same angles, their projections are not proportional.

5. If a circle on one of the trihedral faces be projected, its projection is an ellipse. Its major axis is equal to the diameter of the circle, and parallel to the trace of that face on the plane of projection: that is, in this case, perpendicular to the projection of the third edge of the trihedral angle. For instance, if the circle be on the plane ASB, the major axis of the ellipse will be perpendicular to CD; and the minor axis parallel to CD.

To find the length of the minor axis, let us recur to the first figure. The diameter of the circle which is projected into the minor axis is parallel to SE; and hence its projection is proportional to that of SE, by the first principles of projection. That is, diameter : its projection : : SE : ED : : SA : SD : : $Sa : Sd'$, the last ratio being taken in the second figure. The two diameters of the ellipse are therefore given in magnitude and position; and hence the ellipse itself can be drawn.

6. All circles on these planes will be projected into *similar ellipses.* A simple instrument adapted to the construction of this particular ellipse, varying only in magnitude, would be a desirable addendum to the draughtsman's apparatus. None such has, however, been proposed; although more than one might be suggested, were this the place to do so. The difficulty has been (somewhat clumsily, and very unscientifically) met, by forming a series of elliptic discs to represent the projections of circles of different magnitudes, which are laid upon the paper and traced round with a pencil.* Independently of its so rarely happening that the disc shall be formed to the exact dimensions wanted, there is great difficulty (indeed it is almost impossible) in placing the disc in its strictly correct position.

We come, lastly, to the *practice of isometric projection.*

1. The position of the point to be projected must be given by reference to the three faces of the trihedral angle or coordinate planes; that is, its distances from them respectively. This, of course, is only to be actually obtained by measurement or by constructions or calculations founded on measurements. If, however, the representation be one of a

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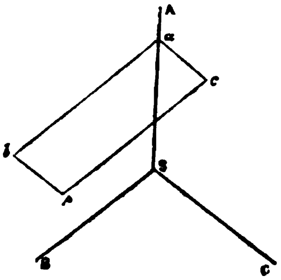
* A series of scales have been published by Mr. Sopwith, of Newcastle, which are adapted to the operations of isometric projection, as it is commonly practised.

design, these measurements are hypothetical, and included in the design. Still in all cases these are data essential to the problem, let its origin and object be what they may.



2. Conceive the figure to represent the point *p* in its actual position as respects the coordinate planes; that is, to the trihedral angle at S. Make S*a* equal to one of the coordinates of *p*, and draw *ac*, *ab* parallel to SC, SB equal to the other two respectively, and join *pb*, *pc*. Then *abpc* is a rectangle, having its sides parallel to a face of the trihedral angle. It will therefore be projected into an isometric parallelogram.

Next, in a second figure (prepared by drawing the three lines SA, SB, SC as thus described), take S*a* equal to the *isometric* magnitude of S*a* in this; draw *ab*, *ac* parallel to SB, SC; make *ab*, *ac* the *isometric* magnitudes of the former ones; and complete the parallelogram *abpc*. Then *p* is the projection required.

Proceeding thus with point after point, till all essential ones are inserted in the drawing, and these being joined by lines or curves (as the case may require), the operation will be completed. Thus the projection of a line is found from those of its two extremities, of a triangle from those of its three angular points, and so on.

SCHOLIUM. There are very few circumstances in this projection that contribute to facility of operation. We have no scale, for instance, by which to measure the projection of any line, except it be parallel to the isometric axis; nor, indeed, can we assign the real length of any of the lines in the projection (the one case excepted) without a long and complicated construction or calculation adapted specifically to each individual case. The real magnitude of an angle on the projection requires a process still more complex. These facts deteriorate much from the value of the method.

On the other hand, when there are several objects to be represented, which are all equal and parallel (as rafters, etc.), the determination of one upon the drawing renders that of the others very easy and simple. In short, the method is adapted to *representation* only; and for this purpose should it alone be used—not for aiding in any ulterior investigations. In the case of machinery, too, as the wheels are mostly parallel to one or other of the three trihedral planes, the method would be very appropriate, had we any good instrument for tracing the ellipses, and likewise of dividing them to correspond with equal divisions of the original circles.

PROJECTIONS OF THE SPHERE.

Before astronomers had arrived at a tolerably full development of the theories of plane and spherical trigonometry, they had recourse to instrumental solutions of the various problems, by means of the projections of the circles of the sphere on a plane surface. Such processes, however, could at best afford but very rude approximations to the true solution, and they are totally unfitted to the modern wants of the science. Even where a rough approximation, as an initial step towards a more accurate solution, or as a general check upon numerical accuracy *ab initio*, is found to be desirable, it is more readily and with less trouble obtained by means of an "artificial globe." For all such purposes, therefore, solutions by projection become useless.

Maps, however, which are representations in plans of points on a spherical surface are still objects of convenience and even of necessity, both as regards the earth and the stars. This renders it necessary to understand the general principles of these projections; but it would be foreign to the purpose of this part of the work to enter into any applications of them; the construction of maps is, moreover, described in the third volume under the head of "Geodesy."

For the definitions, general reference may be made to the spherical geometry.

I. ORTHOGRAPHIC PROJECTIONS OF THE SPHERE.

In this case the projection is made on the plane of one of the great circles of the sphere; usually the equator or one of the meridians. Whatever be the plane chosen, the projections all agree in this; that the projection of a circle is an ellipse, a circle and a diameter being extreme cases of the elliptic projections.

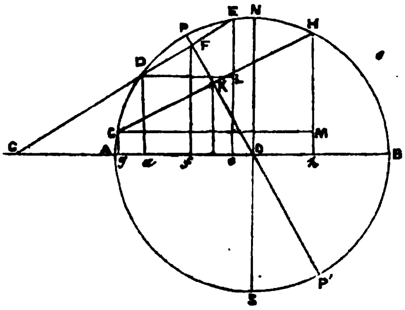
1. The projections of all circles, the planes of which are parallel to the plane of projection are circles, and concentric with that upon the plane of which the projection is made.

2. The projections of all circles, the planes of which are perpendicular to the plane of projection are straight lines, equal respectively to the diameters of those circles. If they be great circles, these lines will be the diameters of the circle upon whose plane the circles are projected; but if less circles, they will be chords of that.

These are too obvious to create difficulty or require further explanation.

3. The projections of circles on planes oblique to the plane of projection are ellipses (*Prop.* IV., p. 252); and the projections on all parallel sections (and of sections equally inclined to the plane of projection) are similar ellipses, having their centres in one straight line, their major axes perpendicular to that line, and equal to the diameter of the projected circle.

Moreover, the major axis is parallel to the line in which the figure-plane intersects the plane of projection; that it is equal to the diameter of the projected circle; and that all circles in the same plane are projected into similar ellipses, having their major axes parallel to the said line of intersection.



Let the annexed figure represent the section of the sphere in the circle ANBS (the section being made by a plane perpendicular to the intersection of the projected circle and plane of projection); DE and GH two parallel sections; PO the radius of the sphere perpendicular to the plane of these circles, and the other lines parallel and perpendicular to AB as indicated in the figure.

Then DE and GH are the diameters of the parallel circles which are projected into the minor axes of the ellipses. The two triangles DLE, GMH which have their corresponding sides parallel are hence similar; and we have

$$DE : DL :: HG : GM.$$

But DE and HG are equal to the major axes of the ellipses, and DL and GM to the minor, viz., to *de* and *gh*. Whence the ellipses are similar.

All the centres of the parallel circles are in OP, and hence their projections in that of OP: that is, in the diameter AB.

4. As a trigonometrical expression, if ϕ denote the arc intercepted between the pole of the plane of projection and that of the circle to be projected, we have

$$\text{major axis} : \text{minor axis} :: 1 : \cos\phi;$$

(for ϕ is also the inclination of the two planes), or if a be the major axis, then the minor is $b = a \cos\phi$.

Scholium. For a small distance round the pole of the plane of projection, the distortion of the spherical figure is not considerable; but beyond a distance of 30° it becomes large; and ultimately the whole becomes too much crowded for distinct delineation.

II. STEREOGRAPHIC PROJECTION OF THE SPHERE.

In this the projecting point is at one extremity of the diameter of the sphere, and the plane of projection is the tangent plane at the other extremity. Some authors project on the diametral plane of the sphere, which is parallel to that tangent plane. The only difference is that the tangent plane is somewhat more convenient, and the figures double the magnitude of those used in the other mode.

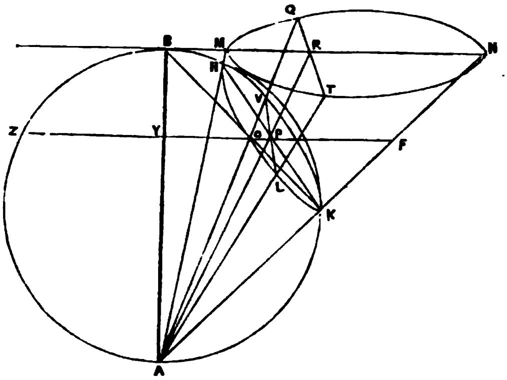
(1.) *All circles parallel to the plane of projection are projected into concentric circles, having the point of contact of the plane and sphere for their common centre.*

(2.) *All circles, great or small, which pass through the projecting point are projected into infinite straight lines; viz., into the intersections of the planes of the circles with the plane of projection.*

(3.) *All circles whatever which are not included in the preceding are projected into circles.*

The demonstration of the properties (1) and (2) may be left as an exercise for the student; (3) is proved in the following manner:—

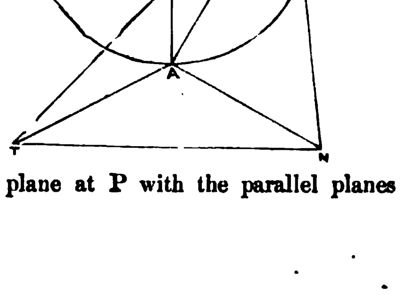
Let a plane be drawn through the centre P of the circle to be projected and the diameter AB; let it cut the plane of projection in BN, the edges of the projecting cone in AK and AH, and the plane of the given circle in HK: draw FY through the center P perpendicular to AB.



Then, since AKB and FYA are right angles and the angle BAK common to the two triangles, we have the angles AFY, KBA also equal. But KBA = KHA (*Euc.* III. 20), and hence AHK is equal to AFY, or ANM. Wherefore since the section HK is a circle, and the angle AHK equal ANM, the section MN is also a circle (*Pls. Chap.* VII. 5).

(4.) *The tangent of an arc and its projection are equal.*

Draw the tangent-planes at A, B, P; and let OQ, TN be the intersections of the plane at P with the parallel planes which touch the



sphere at A and B. Draw APR, and let PQ be the tangent of the given arc. Draw QPT, meeting TN in T. Then TA, TP are tangents to the sphere at P, drawn from T, they are, therefore, equal; and consequently by the similar triangles APT, RPQ, we have RQ = QP. Which is the enunciated property.

(5.) *The projection of the angle formed by the tangents of two circles at their point of intersection is equal to the angle made by the tangents themselves.*

(Same figure.) Let NPO be the tangent to the second arc, produced as before to meet the plane of projection in O; and join OR, OQ, NA, NT.

Then, as in the preceding theorem, OR = OP. Whence the two triangles ORQ, OPQ have the sides OR, RQ equal to OP, PQ and the base OQ common; and therefore the angle ORQ is equal to OPQ, as was enunciated.

SCHOLIUM. A few other properties of this projection possess some elegance, but they are not easy to enunciate in a brief form. Those in (3, 4, 5) are, however, the basis of the system, and the others easily flow from them. At the same time, the expressions of those properties take a much neater form in a symbolical system of investigation; and they are thus given by the best writers—Puissant (in his *Géodésie*), for instance.

III. THE GNOMONIC PROJECTION OF THE SPHERE.

In this the projecting point is the centre of the sphere and the plane of projection a tangent plane. The projecting cones of all less circles are all right cones; and the projections themselves are any of the conic sections indiscriminately; whilst all great circles are projected into straight lines, viz., those in which the planes of the circles cut the tangent-plane.

The properties of this projection are in no respect remarkable; and it is rarely used in the construction of any maps, whether of the earth or the stars. Its only real use is in the construction of dials, which being now a matter of mere curiosity, it would be superfluous to give a further description of the method here.

The application of it in the case of great circles, or of these combined with one less circle, is, however, often used with much advantage in deducing spherical theorems from previously demonstrated plane ones. For this purpose it has been a good deal used by English geometers.

There are several modes of making maps upon the compensation principle, which are adverted to in the article on Geodesy in the third volume.

LONDON:
Printed by W. Clowes & Sons, Stamford Street,
For Her Majesty's Stationery Office.

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